THE CAUCHY-SZEGŐ PROJECTION FOR DOMAINS IN $\mathbb{C}^n$ WITH MINIMAL SMOOTHNESS

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Abstract. We prove $L^p(bD)$-regularity of the Cauchy-Szegő projection[1] for bounded domains $D \subset \mathbb{C}^n$ whose boundary satisfies the minimal regularity condition of class C^2, together with a naturally occurring notion of convexity.

1. Introduction

The purpose of this paper is to prove the L^p-regularity of the Cauchy-Szegő projection[1] for domains $D \subset \mathbb{C}^n$ with $n \geq 2$, whose boundary is subject to minimal smoothness hypotheses. In recent years, the study of the basic domain operators, such as the Cauchy integral and the Bergman projection, has been undertaken in the context of minimal smoothness. However, the corresponding question for the Cauchy-Szegő projection, which raises a number of serious different issues, has hitherto not been broached.

1.1. Background.

The Cauchy-Szegő projection $\mathcal{S}$ is defined as the orthogonal projection of $L^2(bD)$ onto the holomorphic Hardy space $\mathcal{H}^p(bD)$, which is the closure of the subspace of functions on bD that arise as restrictions of functions continuous on $\overline{D}$ and holomorphic in D. A list of some earlier relevant papers includes [PS], [KS-2], [AS-1], [BoLo], [NRSW], [Cu], [H], [FH-1], [FH-2], [MS-2], [CD], [Ko-1], [HNW].

• Let us first recall some relevant facts for $\mathbb{C}^n$ when $n = 1$ (the planar setting). In the special case when D is the unit disc, $\mathcal{S}$ is in

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[1] also known as *Szegő projection*.

fact the Cauchy integral, and its L^p estimate is the classical theorem of M. Riesz. For more general $D \subset \mathbb{C}$ the matter can be briefly put as follows: the lower limit of smoothness of bD that can assure the L^p-boundedness of $\mathcal{S}$ is of one order of differentiability. More precisely, if bD is of class C^1, then $\mathcal{S}$ is bounded on $L^p(bD)$, for $1 < p < \infty$. However if bD is merely Lipschitz, then the conclusion holds for a range $p_D < p < p'_D$, where p_D depends on the Lipschitz bound of D, and in any case $1 < p_D < 4/3$. There are two ways of achieving these results. The first is by conformal mapping (hence essentially for simply connected domains) see [S], [B], [LS-1] and references therein. The second depends on the L^p-theory of the Cauchy integral $\mathbf{C}$ and its boundary transform $\mathcal{C}$, see [C] and [CMM], and proceeds via the identity

$$\mathcal{C} = \mathcal{S}(I - \mathcal{A}), \quad \text{with} \;\; \mathcal{A} = \mathcal{C}^* - \mathcal{C}\,, \tag{1.1}$$

where $\mathcal{C}^*$ is the adjoint of $\mathcal{C}$ on $L^2(bD)$, and I is the identity operator. The issue then becomes the possible invertibility of $I - \mathcal{A}$ in $L^2(bD)$, see [KS-1], [LS-1].

• Turning to the case when $D \subset \mathbb{C}^n$ with $n > 1$, one is immediately faced with several obstacles not seen when $n = 1$.

(a) The fact that the requirement of pseudo-convexity of the domain D must necessarily arise. Since pseudo-convexity is a notion essentially bearing on the second fundamental form of bD, it is reasonable to expect minimal smoothness to be "near C^2", as opposed to "near C^1" for $n = 1$.

(b) The analogue of the approach via conformal mapping is not viable for multiple reasons, one of which is that holomorphic equivalence of domains in $\mathbb{C}^n$ is highly restrictive when $n > 1$, and therefore not applicable to general classes of domains.

(c) The possible use of an identity like (1.1) is problematic, because when $n > 1$ there are infinitely many Cauchy integrals that present themselves, while no one seems appropriate for a direct use of (1.1) (unless D is in fact relatively smooth [KS-2]).

(d) It would be possible to prove the L^p-boundedness of $\mathcal{S}$ by the Calderòn-Zygmund paradigm if we had a satisfactory description of the kernel of this operator. However, the asymptotic formula of Fefferman which would do this (analogous to his well-known description of the Bergman kernel [F]) requires that the domain be relatively smooth, which is not the case in what follows below.

1.2. **Main result.** To state our result we need to make the definition of the Cauchy-Szegő projection precise. If $d\sigma$ denotes the induced

Lebesgue measure on bD, we write $L^p(bD)$ for $L^p(bD, d\sigma)$. In addition, if $d\omega$ is any measure on bD of the form $d\omega = \omega d\sigma$, where the density ω is a strictly positive continuous function on bD, then one can consider $L^p(bD, d\omega)$, but note that this space contains the same elements as $L^p(bD, d\sigma)$ and the two norms are equivalent. Thus we will continue to denote both of these spaces by $L^p(bD)$. However, the distinction between $L^2(bD, d\sigma)$ and $L^2(bD, d\omega)$ become relevant when defining the Cauchy-Szegő projection, because these spaces have different inner products that give different notions of orthogonality. So the Cauchy-Szegő projection $\mathcal{S}_\omega$ is the orthogonal projection of $L^2(bD, \omega d\sigma)$ onto $\mathcal{H}^2(bD, \omega\, d\sigma)$. Note that $\mathcal{S}_1$ is the "$\mathcal{S}$" discussed above, but there is no simple connection between the general $\mathcal{S}_\omega$ and $\mathcal{S}_1$. Nevertheless, the case when $d\omega$ is the Leray-Levi measure that arises below, is key to understanding the general result. It states:

Suppose $D \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain whose boundary is of class C^2 and is strongly pseudo-convex. Then $\mathcal{S}_\omega$ is a bounded operator on $L^p(bD, d\omega)$, for $1 < p < \infty$.

There are five main steps in the proof.

Step 1: Cauchy integrals. For our purposes a Cauchy integral $\mathbf{C}$ is an operator mapping functions on bD to functions on D given by

$$\mathbf{C}(f)(z) = \int\limits_{w \in bD} f(w)\, C(w, z), \quad z \in D, \tag{1.2}$$

with the following properties:

(i) $\mathbf{C}$ produces holomorphic functions. More precisely, if f is integrable on bD then $\mathbf{C}(f)$ is holomorphic in D.

(ii) $\mathbf{C}$ reproduces holomorphic functions. That is,

$$\mathbf{C}(f)(z) = F(z), \quad z \in D,$$

whenever F is holomorphic in D and continuous on $\overline{D}$, and $F\big|_{bD} = f$.

(iii) The kernel $C(w, z)$ is "explicit" enough to allow "relevant" computations.

A general method for obtaining integrals that satisfy the requirements (ii) and (iii) (but not necessarily (i)) is that of the Cauchy-Fantappiè formalism, see e.g., [LS-3]. The point of departure is a "generating"

form

$$G(w,z)=\sum_{j=1}^{n} G_j(w,z)\,dw_j$$

that satisfies

$$g(w,z)=\langle G(w,z),w-z\rangle \neq 0 \tag{1.3}$$

if $z\in D$ and w is in a small neighborhood of bD. (Here, $\langle G(w,z),w-z\rangle$ denotes the action of $G(w,z)$ on the vector $w-z$.) Then the corresponding Cauchy integral is defined by (1.2), with

$$C(w,z)=\frac{1}{(2\pi i)^n}\frac{G\wedge(\overline{\partial}_w G)^{n-1}}{g(w,z)^n}\,.$$

One observes that when $n=1$, there is only one such integral kernel (namely the familiar Cauchy kernel $C(w,z)=dw/2\pi i(w-z)$), while for $n\geq 2$ there are infinitely many such kernels. Moreover, the existence of a generating form that satisfies property (i.) is closely related (by way of the so-called Levi problem) to the requirement that D be pseudo-convex, and such forms have been constructed only when D is actually strongly pseudo-convex (and relatively smooth), see [H], [R], [KS-2]. These constructions take for $z\in D$ near $w\in bD$ and ρ a defining function of D

$$G(w,z)=\partial\rho(w)-\frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial^2\rho(w)}{\partial w_j\partial w_k}(w_k-z_k)\,dw_j\,, \tag{1.4}$$

and then extend $G(w,z)$ to $z\in D$ by differing methods. It is to be noted that the function $g(w,z)$ that results from this choice of $G(w,z)$ via the construction (1.3) is the Levi polynomial at $w\in bD$.

Now if we take (1.4) as our starting point we immediately run into a first obstacle. Since we have assumed that D (that is the function ρ) is of class C^2, the denominator g^n in $C(w,z)$ above, cannot be guaranteed any degree of smoothness in w, beyond continuity in that variable. But all known methods of proving L^2 (or L^p) boundedness of singular integrals (that is, the $T(1)$ theorem and its variants) require some degree of smoothness of the kernel away from the diagonal $\{w=z\}$. One way to get around this difficulty is to replace the matrix $\{\partial^2\rho/\partial w_j\partial w_k\}$ appearing in (1.4) by a C^1-smooth matrix (as is done in [Ra]). However the Cauchy integral constructed this way would not be of any substantial use to us. What we require below is a *family* of Cauchy integrals $\{\mathbf{C}_\epsilon\}_\epsilon$, where for each ϵ we have replaced $\{\partial^2\rho/\partial w_j\partial w_k\}$ by an appropriate C^1-smooth matrix $\{\tau^\epsilon_{jk}\}$ with uniform error on bD (less than ϵ), and g is replaced by the corresponding $\{g_\epsilon\}_\epsilon$. However this

approximation comes at a price, which increases as $\epsilon \to 0$, as we will see below.

Step 2: $L^p(bD)$-regularity of the $\mathcal{C}_\epsilon$. We apply the machinery of the general $T(1)$ theorem to to the operators $\mathcal{C}_\epsilon$, which are the boundary restrictions of the Cauchy integrals $\mathbf{C}_\epsilon$, and to do this requires four things. First, we need a space of homogeneous type that reflects the non-isotropic geometry of bD. Second, we need to establish the difference inequalities for the kernel of $\mathcal{C}_\epsilon$, and these can be achieved because we have replaced the matrix $\{\partial^2\rho/\partial w_j \partial w_k\}$ by a suitable approximation $\{\tau^\epsilon_{jk}\}_\epsilon$. The third item is the analysis of the formal adjoint of $\mathcal{C}_\epsilon$ with respect to the inner product of $L^2(bD, d\lambda)$. Here $d\lambda$ is the Leray-Levi measure, defined as integration on bD with respect to the $(2n-1)$-form $\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1}/(2\pi i)^n$. The analysis of such adjoint is done by identifying the "essential part" of $\mathcal{C}_\epsilon$, which is given by $\mathcal{C}^\sharp_\epsilon$, with

$$\mathcal{C}^\sharp_\epsilon(f)(z) = \int\limits_{w \in bD} \frac{f(w)}{g_\epsilon(w,z)^n}\, d\lambda(w)\,.$$

The fourth item is the proof of the required "cancellation" properties of of $\mathcal{C}^\sharp_\epsilon$, and these are expressed as the action of $\mathcal{C}^\sharp_\epsilon$ on "bump functions". For this the key observation is that, unlike the case $n = 1$, whenever $f \in C^1(bD)$, then

$$\mathcal{C}^\sharp_\epsilon(f) = \mathcal{E}_\epsilon(df) + \mathcal{R}^\sharp_\epsilon(f)$$

where both the kernels of $\mathcal{E}_\epsilon$ and $\mathcal{R}^\sharp_\epsilon$ have a singularity weaker by one order than that of $\mathcal{C}^\sharp_\epsilon$. This concludes the proof of the regularity of $\mathcal{C}_\epsilon$ on $L^p(bD, d\lambda)$ for any $1 < p < \infty$. (We will return to the general case: $L^p(bD, \omega d\sigma)$ in Step 5 below.)

Step 3: Relating the Cauchy-Szegő projection to the $\mathcal{C}_\epsilon$'s. At this point we establish an analogous formulation of the original identity (1.1) for the Cauchy-Szegő projection relative to the space $L^2(bD, d\lambda)$, which we denote $\mathcal{S}_\lambda$, namely the identity:

$$\mathcal{S}_\lambda(I - \mathcal{A}_\epsilon) = \mathcal{C}_\epsilon \tag{1.5}$$

where $\mathcal{A}_\epsilon = \mathcal{C}^*_\epsilon - \mathcal{C}_\epsilon$, and * denotes the adjoint in $L^2(bD, d\lambda)$. At this stage we rely on some results on the Hardy space $\mathcal{H}^2(bD)$ which will appear separately in [LS-5].

Step 4: Proving the boundedness of $\mathcal{S}_\lambda$ on $L^p(bD, d\lambda)$. One would then like to invert the operator $I - \mathcal{A}_\epsilon$ that appears in (1.5) by a partial Neumann series, but the fact is that for our C^2-smooth domains, the quantity $\|\mathcal{A}_\epsilon\|$ is unbounded as $\epsilon \to 0$, with $\|\cdot\|$ the operator norm

acting on any $L^p(bD)$ space. This is the price we have to pay for replacing the original matrix $\{\partial^2\rho/\partial w_j\partial w_k\}$ with the smoother matrices $\{\tau^\epsilon_{jk}\}_\epsilon$. To surmount this difficulty we truncate our operator and write for any $s > 0$

$$\mathcal{C}_\epsilon = \mathcal{C}^s_\epsilon + \mathcal{R}^s_\epsilon\,,$$

where $\mathcal{C}^s_\epsilon$ has the same kernel as $\mathcal{C}_\epsilon$, except that it is appropriately cut off to be supported in $\mathtt{d}(w,z) \le s$, where $\mathtt{d}(w,z)$ is the non-isotropic quasi-distance for the space of homogeneous type described in Step 2 above. What this truncation achieves is the following important fact: if s is sufficiently small in terms of ϵ, then

$$\|(\mathcal{C}^s_\epsilon)^* - \mathcal{C}^s_\epsilon\|_{L^p\to L^p} \lesssim \epsilon^{1/2} M_p. \tag{1.6}$$

(Here the adjoint is again taken with respect to $L^2(bD, d\lambda)$, with $d\lambda$ the Leray-Levi measure.) This feature of $\mathcal{C}^s_\epsilon$ allows us to express (1.5) in the equivalent form

$$\mathcal{S}_\lambda = \big(\mathcal{C}_\epsilon + \mathcal{S}_\lambda\big((\mathcal{R}^s_\epsilon)^* - \mathcal{R}^s_\epsilon\big)\big)\,\big(I - \big((\mathcal{C}^s_\epsilon)^* - \mathcal{C}^s_\epsilon\big)\big)^{-1}, \tag{1.7}$$

which is valid for ϵ sufficiently small in terms of p, and for s sufficiently small in terms of ϵ. The complementary fact is that while the quantity $\|(\mathcal{R}^s_\epsilon)^* - \mathcal{R}^s_\epsilon\|_{L^p\to L^p}$ is not small (in fact, in general this is unbounded as $\epsilon \to 0$), one has the redeeming property that each of $\mathcal{R}^s_\epsilon$ and $(\mathcal{R}^s_\epsilon)^*$ maps: $L^1(bD, d\lambda)$ to $L^\infty(bD)$ (in fact to $C(bD)$) for each ϵ and s. Taking all this into account we conclude from (1.7) that $\mathcal{S}_\lambda$ is bounded on $L^p(bD, d\lambda)$ for each $1 < p < \infty$.

Step 5: Passage to $\mathcal{S}_\omega$ for general ω. First we note that the boundedness of the $\mathcal{C}_\epsilon$ on $L^p(bD, d\lambda)$ immediately gives their boundedness on $L^p(bD, \omega d\sigma)$ via the remarks that we made before. While the corresponding result for the Cauchy-Szegő projections cannot be obtained in the same way, the main idea for $\mathcal{S}_\omega$ is as follows. If † denotes the adjoint with respect to the inner product of $L^2(bD, \omega d\sigma)$, and if we write $d\omega = \varphi\, d\lambda$, then the fact that $(\mathcal{C}^s_\epsilon)^\dagger = \varphi(\mathcal{C}^s_\epsilon)^*\varphi^{-1}$, and the continuity of φ, allow us to obtain from (1.6) that

$$\|(\mathcal{C}^s_\epsilon)^\dagger - \mathcal{C}^s_\epsilon\|_{L^p\to L^p} \lesssim \epsilon^{1/2} M_p. \tag{1.8}$$

With these things in place one can then proceed as in Step 5.

It is worthwhile to point out that the original Cauchy-Szegő projection $\mathcal{S}_1$ (defined with respect to the induced Lebesgue measure $d\sigma$) is included here, because it turns out that $d\sigma = \varphi_0\, d\lambda$ where the density φ_0 is continuous and positive on account of the strong pseudo-convexity and C^2-regularity of D.

1.3. **Organization of the paper.** This paper consists of two parts. Part I deals with the Cauchy integrals, and it includes sections 2 through 4. Sections 2 and 3 contain a review of preliminary facts that are needed. Since in the main these were also used in our work on the Bergman projection [LS-2] and the Cauchy-Leray integral [LS-3] and [LS-4], the details of the proofs are for the most part omitted. Section 4 contains the L^p-theory of the Cauchy integrals $\mathbf{C}_\epsilon$ and the corresponding boundary transforms $\mathcal{C}_\epsilon$.

In Part II we turn to the Cauchy-Szegő projection. In Section 5 we prove the key fact (1.6) (Proposition 18), which proves the main result in the context of the Leray-Levi measure (Theorem 15). Section 6 then concludes by dealing with the general case (Theorem 16). The argument deducing (1.8) from (1.6) rests on a general result (Lemma 24) involving operators whose kernels have support appropriately close to the diagonal.

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Part I: Cauchy Integrals

In this first part we undertake the study of a family $\{\mathbf{C}_\epsilon\}$ of Cauchy integrals that depend on a parameter ϵ, and which are constructed using the Cauchy-Fantappiè formalism. Here we focus on the properties of $\mathbf{C}_\epsilon$ for fixed ϵ. What happens when ϵ varies, in particular the behavior of $\mathbf{C}_\epsilon$ as $\epsilon \to 0$, will be studied in Part II, where this will play a key role in the understanding of the Cauchy-Szegő projection.

2. The fundamental denominators g_0 and g_ϵ

A number of results needed below that are known (see [LS-3], [LS-4], [Ra]) are restated here without proof. An exception is Proposition 1 and its corollary.

2.1. **The functions g_0 and g_ϵ.**

We consider a bounded domain D in $\mathbb{C}^n$ with defining function $\rho : \mathbb{C}^n \to \mathbb{R}$ of class C^2, for which $D = \{\rho < 0\}$, and $|\nabla\rho| > 0$ where $\rho = 0$. We assume that ρ is strictly plurisubharmonic. The assumptions regarding the domain D and ρ will be in force throughout, and so will not be restated below.

We let $\mathcal{L}_0(w,z)$ be the negative of the Levi polynomial at $w \in bD$, given by

$$\mathcal{L}_0(w,z) = \langle \partial\rho(w), w-z\rangle - \frac{1}{2}\sum_{j,k}\frac{\partial^2\rho(w)}{\partial w_j \partial w_k}(w_j - z_j)(w_k - z_k)$$

where $\partial\rho(w)$ is the 1-form $\sum \rho_{w_j}(w)dw_j$, and the expression $\langle\partial\rho(w), w-z\rangle$ denotes the action of $\partial\rho(w)$ on the vector $w-z$, that is

$$\langle\partial\rho(w), w-z\rangle = \sum_j \frac{\partial\rho(w)}{\partial w_j}(w_j - z_j)\,.$$

The strict plurisubharmonicity of ρ implies that

$$2\,\mathrm{Re}\,\mathcal{L}_0(w,z) \geq -\rho(z) + c|w-z|^2, \quad \text{for some } c>0,$$

whenever $w \in bD$ and $z \in \overline{D}$ is sufficiently close to w. To ensure that this inequality may hold globally we make our first modification of $\mathcal{L}_0$ and replace it with g_0 given as

$$g_0(w,z) = \chi\mathcal{L}_0 + (1-\chi)|w-z|^2.$$

Here $\chi = \chi(w,z)$ is a C^∞ cut-off function with $\chi = 1$ when $|w-z| \leq \mu/2$ and $\chi = 0$ if $|w-z| \geq \mu$. Then if μ is chosen sufficiently small (and then kept fixed throughout) we have that

$$\mathrm{Re}\,g_0(w,z) \geq c\,(-\rho(z) + |w-z|^2) \tag{2.1}$$

for z in $\overline{D}$ and w in bD, with c a positive constant.

The modified Levi polynomial g_0 is not yet quite right for our needs, because in general it has no smoothness beyond continuity in the variable w. So for each $\epsilon > 0$ we consider a variant g_ϵ defined as follows. We find an $n \times n$ matrix $\{\tau^\epsilon_{jk}(w)\}$ of C^1 functions so that

$$\sup_{w\in bD}\left|\frac{\partial^2\rho(w)}{\partial w_j \partial w_k} - \tau^\epsilon_{jk}(w)\right| \leq \epsilon, \quad \text{for} \quad 1 \leq j,k \leq n\,.$$

We then set

$$\mathcal{L}_\epsilon(w,z) = \langle\partial\rho(w), w-z\rangle - \frac{1}{2}\sum_{j,k}\tau^\epsilon_{jk}(w)\,(w_j-z_j)(w_k-z_k)\,,$$

and define

$$g_\epsilon(w,z) = \chi\mathcal{L}_\epsilon + (1-\chi)|w-z|^2 \quad \text{for } z\,, w \in \mathbb{C}^n\,. \tag{2.2}$$

Now g_ϵ is of class C^1 in w (it is of class C^∞ in z). We note that

$$|g_0(w,z) - g_\epsilon(w,z)| \lesssim \epsilon|w-z|^2 \quad \text{for } w \in bD\,,$$

and hence if ϵ is taken sufficiently small (in terms of the constant c appearing in (2.1)) then automatically

$$\mathrm{Re}\,g_\epsilon(w,z) \geq c'(-\rho(z) + |w-z|^2), \quad \text{for } z \in \overline{D},\ w \in bD\,, \tag{2.3}$$

for an appropriate positive c'. There is also the variant

$$\mathrm{Re}\,g_\epsilon(w,z) \geq c'(\rho(w) - \rho(z) + |w-z|^2)$$

for z and w in a neighborhood of bD. We shall always assume that ϵ is restricted to be so small that (2.3) holds. As a direct consequence of (2.1) we then also have

$$|g_\epsilon(w,z)| \approx |g_0(w,z)| \tag{2.4}$$

where the constants implied in the inequality $\lesssim$ above, and the equivalence $\approx$, are independent of ϵ.

2.2. **Special coordinate system.** To obtain a better understanding of g_0 and g_ϵ we introduce for each $w \in bD$ a special coordinate system centered at w. We let ν_w denote the inner unit normal at w, so $\nu_w = -\nabla\rho(w)/|\nabla\rho(w)|$. We set $e_n = i\nu_w$, and take $\{e_1, \dots, e_{n-1}, e_n\}$ to be an orthonormal basis of $\mathbb{C}^n$. Our coordinates of a point $z \in \mathbb{C}^n$ are then determined by

$$z - w = \sum_j z_j e_j \, .$$

Note that then the coordinate $z_n = x_n + iy_n$ is intrinsically determined, as well as the length of the orthogonal complement,

$$|z'| = \left(\sum_{j=1}^{n-1} |z_j|^2 \right)^{1/2} .$$

Using the fact that $2\,\partial\rho(w) = -i|\nabla\rho(w)|dw_n$, we see that

$$\langle \partial\rho(w), w - z \rangle = -\frac{i}{2}|\nabla\rho(w)|\, z_n$$

(see also [LS-4]). Looking back at (2.2) we then have

$$g_\epsilon(w,z) = -ic_w\, z_n + Q(z) \tag{2.5}$$

where $c_w = |\nabla\rho(w)|/2$, and $Q(z) = Q^\epsilon_w(z)$ is a homogeneous quadratic polynomial in $z_1, \dots, z_n$. The only property of Q that we need to know is that $|Q(z)| \le c|z|^2$, with a constant c independent of ϵ and w.

The main estimates for $g_\epsilon(w,z)$ (and hence also for $g_0(w,z)$) are contained in the following

Proposition 1. *For each $w \in bD$, we have*

(i) $|g_\epsilon(w,z)| \approx |x_n| + |w - z|^2 + |\rho(z)|$, *for $z \in \overline{D}$.*

In particular, we have

(ii) $|g_\epsilon(w,z)| \approx |x_n| + |z'|^2$, *for $z \in bD$.*

The constants implicit in these equivalences are independent of ϵ, w and z.

Proof. We begin by observing that $z = w$ we have

$$\frac{\partial\rho}{\partial z_j} = 0, j = 1, \ldots, n-1; \quad \frac{\partial\rho}{\partial x_n} = 0; \quad \frac{\partial\rho}{\partial y_n} = -|\nabla\rho(w)|.$$

Thus, by Taylor's theorem,

$$\rho(z) = -|\nabla\rho(w)|\, y_n + O(|z-w|^2), \quad \text{and therefore} \tag{2.6}$$

$$|y_n| \lesssim |\rho(z)| + |w-z|^2.$$

Combining this with (2.5) and the comments thereafter gives

$$|\operatorname{Re} g_\epsilon(w,z)| \lesssim |\rho(z)| + |w-z|^2.$$

However when $z \in \overline{D}$ we have $-\rho(z) \geq 0$, and so (2.3) grants the opposite inequality, which yields

$$\operatorname{Re} g_\epsilon(w,z)| \approx |\rho(z)| + |w-z|^2. \tag{2.7}$$

Moreover (2.5) immediately gives

$$\operatorname{Im} g_\epsilon(w,z) = -c_w\, x_n + O(|w-z|^2), \quad \text{which leads to}$$

$$\left\{\begin{array}{rcl} |\operatorname{Im} g_\epsilon(w,z)| & \lesssim & |x_n| + |w-z|^2, \quad \text{and} \\ \\ |x_n| & \lesssim & |\operatorname{Im} g_\epsilon(w,z)| + |w-z|^2. \end{array}\right. \tag{2.8}$$

A combination of (2.8) with (2.7) then implies conclusion (i).

The second conclusion follows from the first because $z \in bD$ implies $\rho(z) = 0$, while $|w-z|^2 = |z_n|^2 + |z'|^2 = x_n^2 + y_n^2 + |z'|^2$, and $x_n^2 \approx |x_n|$ if x_n is bounded, while by (2.6), $|y_n| \lesssim |w-z|^2$, if $z \in bD$. □

There are two consequences that can be drawn from Proposition 1. First, if z and w are both in bD then

$$|g_\epsilon(w,z)| \approx |\operatorname{Im}\langle\partial\rho(w), w-z\rangle| + |w-z|^2. \tag{2.9}$$

In fact, by the above

$$\frac{\partial\rho(w)}{\partial z_n} = \frac{i}{2}|\nabla\rho(w)|, \quad \text{and hence } \langle\partial\rho(w), w-z\rangle = -ic_w z_n,$$

with $c_n > 0$ as in (2.5); so $\operatorname{Im}\langle\partial\rho(w), w-z\rangle = -c_w x_n$. Thus (2.9) is a consequence of the first conclusion of Proposition 1.

Our next assertion is the analogue of [LS-4, Lemma 4.3]. For $z \in bD$ we write $z^\delta = z + \delta\nu_z$, where ν_z is the inward unit normal at z.

Corollary 2. *For $\delta > 0$ sufficiently small, we have*

$$|g_\epsilon(w, z^\delta)| \approx |g_\epsilon(w,z)| + \delta, \quad \textit{for } w, z \in bD, \textit{ and } z^\delta \textit{ as above.}$$

Proof. It suffices to prove this when $|w-z| \leq c_1$, where c_1 is a small positive constant to be chosen below. The result for $|w-z| \geq c_1$ is a trivial consequence of conclusion (i) in Proposition 1.

Now let $z_n = x_n + iy_n$ be the n-th coordinate of z in the coordinate system centered at w that was chosen earlier, and $z_n^\delta = x_n^\delta + iy_n^\delta$ be the corresponding coordinate of $z^\delta = z + \delta\nu_z$. Then, letting $(\cdot,\cdot)$ denote the hermitian inner product in $\mathbb{C}^n$, we have

$$z_n = (z-w, e_n),\ \ z_n^\delta = (z^\delta - w, e_n),\ \text{ and therefore}$$

$z_n^\delta = z_n + \delta(\nu_z, e_n) = z_n + \delta(\nu_w, e_n) + O(\delta|z-w|),$ while $(\nu_w, e_n) = ic_w$. Hence

$$x_n^\delta = x_n + O(\delta|z-w|). \tag{2.10}$$

Also, by Taylor's theorem $\rho(z^\delta) = -|\nabla\rho(z)|\delta + O(\delta^2)$ as $\delta \to 0$ since $\rho(z) = 0$ and $(\nu_z, \nabla\rho(z)) = -|\nabla\rho(z)|$. Thus

$$|\rho(z^\delta)| \approx \delta \quad \text{for small } \delta > 0. \tag{2.11}$$

Finally, we note that $|z^\delta - w|^2 = |z-w|^2 + O(\delta|z-w| + \delta^2)$, and applying this along with (2.10) and (2.11) via conclusion (i), we see that

$$|g_\epsilon(w, z^\delta)| \approx |x_n^\delta| + |z^\delta - w|^2 + |\rho(z^\delta)| \approx |x_n| + |z-w|^2 + \delta + O(\delta|z-w| + \delta^2).$$

Now we merely need to take c_1 and δ sufficiently small to absorb the "O" term above into δ, to get

$$|g_\epsilon(w,z)| \approx |x_n| + |w-z|^2 + \delta\,,$$

and thus using conclusion (i) again proves the corollary. □

Note that in view of (2.4) the conclusions of Proposition 1 and Corollary 2 hold for g_0 as well as g_ϵ.

2.3. **Geometry of the boundary of D.** We define the function $\mathtt{d}(w,z)$ by

$$\mathtt{d}(w,z) = |g_0(w,z)|^{1/2}.$$

Note that by (2.4) we also have

$$\mathtt{d}(w,z) \approx |g_\epsilon(w,z)|^{1/2} \quad \text{for all } \ \epsilon. \tag{2.12}$$

For $w, z \in bD$, $\mathtt{d}(w,z)$ has the properties of a quasi-distance, namely

Proposition 3. *For $\mathtt{d}(w,z)$ defined as above, we have*

(a) $\mathtt{d}(w,z) \geq 0$, *and* $\mathtt{d}(w,z) = 0$ *only when* $w = z$.

(b) $\mathtt{d}(w,z) \approx \mathtt{d}(z,w)$

(c) $\mathtt{d}(w,z) \lesssim \mathtt{d}(w,\zeta) + \mathtt{d}(\zeta,z)$

whenever $w, \zeta, z \in bD$.

Proof. Conclusion (a) is obvious from (2.9) and the definition of $\mathtt{d}(w, z)$. Next, observe that

$$|\operatorname{Im}\langle \partial\rho(w), w - z\rangle| = |\operatorname{Im}\langle \partial\rho(z), z - w\rangle| + O(|w - z|^2).$$

Therefore by (2.9), we have that $|g_\epsilon(w, z)| \approx |g_\epsilon(z, w)|$, which implies conclusion (b). Finally, note that

$$\langle \partial\rho(w), w - z\rangle - \langle \partial\rho(w), w - \zeta\rangle = \langle \partial\rho(w), \zeta - z\rangle$$

and the latter equals $\langle \partial\rho(\zeta), \zeta - z\rangle + O(|\zeta - z|\,|w - \zeta|)$ since we have that $\partial\rho(w) - \partial\rho(\zeta) = O(|w - \zeta|)$. It follows that $|\operatorname{Im}\langle \partial\rho(w), w - z\rangle|$ is bounded above by

$$|\operatorname{Im}\langle \partial\rho(w), w - \zeta\rangle| + |\operatorname{Im}\langle \partial\rho(\zeta), \zeta - z\rangle| + O(|w - \zeta|^2 + |z - \zeta|^2).$$

Also, $|w - z|^2 \lesssim |w - \zeta|^2 + |\zeta - z|^2$. From these observations and (2.9) we obtain

$$|g_0(w, z)| \lesssim |g_0(w, \zeta)| + |g_0(\zeta, z)|,$$

which grants conclusion (c). $\square$

A final, simple observation about the quasi-distance $\mathtt{d}$ is that

$$|w - z| \lesssim \mathtt{d}(w, z) \lesssim |w - z|^{1/2}, \quad w, z \in bD, \tag{2.13}$$

which follows immediately from (2.9) via the Cauchy-Schwartz inequality, and the definition of $\mathtt{d}$.

At this stage it is worth recording the following facts proved in the same spirit as the proof of Proposition 3.

For $A(w, z)$ equal to $g_\epsilon(w, z)$ we have

$$|A(w, z) - A(w', z)| \ \le\ c_\epsilon\big(\mathtt{d}(w, w')^2 + \mathtt{d}(w, w')\,\mathtt{d}(w, z)\big) \tag{2.14}$$

and for $A(w, z)$ equal to either $g_\epsilon(z, w)$ or $\operatorname{Im}\langle \partial\rho(w), w - z\rangle$

$$|A(w, z) - A(w', z)| \ \lesssim\ \mathtt{d}(w, w')^2 + \mathtt{d}(w, w')\,\mathtt{d}(w, z) \tag{2.15}$$

where the implicit constant in the second of these inequalities does not depend on ϵ. We prove the first inequality; the inequality for $A(w, z) = g_\epsilon(z, w)$ or $\operatorname{Im}\langle \partial\rho(w), w - z\rangle$ will follow by a similar argument. We see from (2.2) that

$$g_\epsilon(w, z) = \langle \partial\rho(w), w - z\rangle + Q_w(w - z), \quad \text{if} \quad |w - z| \le \mu/2$$

where $Q_w(u)$ is a quadratic form in u. Thus we may split the difference $g_\epsilon(w, z) - g_\epsilon(w', z)$ as the sum of two terms: $I + II$, where

$$I = \langle \partial\rho(w), w - z\rangle - \langle \partial\rho(w'), w' - z\rangle$$

and
$$II = Q_w(w-z) - Q_{w'}(w'-z).$$
Now
$$\begin{aligned} I &= \langle \partial\rho(w), w-w'\rangle + \langle \partial\rho(w) - \partial\rho(w'), w'-z\rangle = \\ &= g_\epsilon(w,w') - Q_w(w-w') + \langle \partial\rho(w) - \partial\rho(w'), w'-z\rangle\,. \end{aligned}$$

Since $|g_\epsilon(w,w')| \approx \mathtt{d}(w,w')^2$, the identity above and (2.13) give
$$|I| \lesssim \mathtt{d}(w,w')^2 + \mathtt{d}(w,w')\,\mathtt{d}(w,z).$$
On the other hand
$$II \ \leq\ \big|Q_w(w-z) - Q_w(w'-z)\big| \ +\ \big|Q_w(w'-z) - Q_{w'}(w'-z)\big|\,,$$
with
$$|Q_w(w-z) - Q_w(w'-z)| \lesssim \mathtt{d}(w,w')^2 + \mathtt{d}(w,w')\,\mathtt{d}(w,z)\,,$$
whereas
$$|Q_w(w'-z) - Q_{w'}(w'-z)| \ \leq c_\epsilon\,|w-w'|\,|w'-z|^2$$
where
$$c_\epsilon = \sup_{\substack{w\in bD\\ 1\leq j,k\leq n}} |\nabla\tau^\epsilon_{j,k}(w)|,$$
and this grants
$$|Q_w(w'-z) - Q_{w'}(w'-z)| \ \leq\ c_\epsilon\big(\mathtt{d}(w,w')^2 + \mathtt{d}(w,w')\,\mathtt{d}(w,z)\big)\,.$$
This proves (2.14); the proof of (2.15) is similar but does not involve the bound c_ϵ.

We next introduce the Leray-Levi measure $d\lambda$ defined on bD. The proofs of the assertions in the rest of this section follow closely those given in a broadly parallel situation in [LS-4, Section 3.4] and so details will be omitted. The Leray-Levi measure $d\lambda$ on bD is defined by the linear functional
$$f \mapsto \int\limits_{bD} f(w)\,d\lambda(w) = \frac{1}{(2\pi i)^n}\int\limits_{bD} f(w)\,j^*\big(\partial\rho\wedge(\overline{\partial}\partial\rho)^{n-1}\big)\,(w)\,,$$
with ρ our defining function, and where $(\overline{\partial}\partial\rho)^{n-1}$ is the $(n-1)$-fold wedge product of $\overline{\partial}\partial\rho$, and with j^* denoting the pull-back under the inclusion
$$j : bD \hookrightarrow \mathbb{C}^n\,.$$
Then one has
$$d\lambda(w) = (2\pi i)^{-n} j^*\big(\partial\rho\wedge(\overline{\partial}\partial\rho)^{n-1}\big)(w) = \Lambda(w)d\sigma(w), \tag{2.16}$$

where $d\sigma$ is the induced Lebesgue measure, and $\Lambda(w)$ is a continuous function such that

$$c_1 \leq \Lambda(w) \leq c_2 \,, \quad w \in bD \,,$$

with c_1 and c_2 two positive constants. In fact

$$\Lambda(w) = (n-1)!(4\pi)^{-n} |\det \rho(w)| \, |\nabla \rho(w)|, \quad w \in bD,$$

where $\det \rho(w)$ is the determinant of the $(n-1) \times (n-1)$ matrix

$$\left\{ \frac{\partial^2 \rho}{\partial z_j \overline{\partial} z_k} \right\} \Bigg|_{z=w}, \quad 1 \leq j, k \leq n-1,$$

computed in the coordinate system $(z_1, \ldots, z_n)$ centered at w that was introduced above. (See [Ra, Lemma VII.3.9].) The C^2 character of ρ together with its strict plurisubharmonicity then establishes (2.16). The particular relevance of the Leray-Levi measure will become apparent when we consider adjoints of our operators.

Our next assertions concern the boundary balls $\{\mathbf{B}_r(w)\}$ determined via the quasi-distance $\mathtt{d}$ and their measures. We define

$$\mathbf{B}_r(w) = \{ z \in bD \ : \ \mathtt{d}(w, z) < r \}, \quad \text{where } w \in bD.$$

We also consider the "box"

$$\widetilde{\mathbf{B}}_r(w) = \{ z \in bD \ : \ |x_n| < r^2, \ |z'| < r \}, \quad w \in bD.$$

We then have the equivalence $\mathbf{B}_r(w) \approx \widetilde{\mathbf{B}}_r(w)$, which means

$$\widetilde{\mathbf{B}}_{c_r}(w) \subset \mathbf{B}_r(w) \subset \widetilde{\mathbf{B}}_{c_2 r}(w),$$

for two positive constants c_1 and c_2 that are independent of $r > 0$ and $w \in bD$. These inclusions follow directly from conclusion (ii) in Proposition 1 and the fact that

$$|g_0(w, z)| \approx |g_\epsilon(w, z)| \approx \mathtt{d}(w, z)^2 \,.$$

Using the arguments set down in [LS-4, Section 3.5], one can show that the equivalence of $\mathbf{B}_r(w)$ with $\widetilde{\mathbf{B}}_r(w)$ implies that

$$\lambda(\mathbf{B}_r(w)) \approx r^{2n}, \quad 0 \leq r \leq 1,$$

and hence also $\sigma(\mathbf{B}_r(w)) \approx r^{2n}, \quad 0 \leq r \leq 1.$

As a final consequence (as is shown in [LS-4]) we have

$$(2.17) \qquad \int\limits_{w \in \mathbf{B}_r(z)} \mathtt{d}(w, z)^{-2n+\beta} d\lambda(w) \leq c_\beta \, r^\beta; \int\limits_{w \notin \mathbf{B}_r(z)} \mathtt{d}(w, z)^{-2n-\beta} d\lambda(w) \leq c_\beta \, r^{-\beta}$$

for $0 < r < 1$ and $\beta > 0$. When $\beta = 0$ we can assert that

$$\int\limits_{w \notin \mathbf{B}_r(z)} \mathtt{d}(w,z)^{-2n}\, d\lambda(w) \ \le\ c\,\log(1/r), \quad \text{for}\ \ 0 < r < 1/2. \tag{2.18}$$

3. A family of Cauchy integrals: definition, correction and initial properties

Here we define the Cauchy integrals $\{\mathbf{C}_\epsilon\}_\epsilon$ (determined by the the denominators $\{g_\epsilon\}_\epsilon$) and study their properties when ϵ is kept fixed; for convenience of notation we will henceforth drop explicit reference to ϵ and will resume doing so in Part II, when the dependence on ϵ will again be relevant (in fact crucial). Thus, for the time being we will write g for g_ϵ; $\mathbf{C}$ for $\mathbf{C}_\epsilon$, and so forth.

3.1. **A Cauchy-Fantappié integral.** Our Cauchy integral will be defined as the sum of two operators. The first, $\mathbf{C}^1$, is a Cauchy-Fantappié integral. To describe it we first isolate a 1-form G closely related to the denominator g.

We set

$$G(w,z) = \tag{3.1}$$

$$= \chi\left[\partial\rho(w) - \frac{1}{2}\sum_{j,k}\tau^{\epsilon}_{j,k}(w)\,(w_j - z_j)\,dw_k\right] + (1-\chi)\sum_k(\overline{w}_k - \overline{z}_k)\,dw_k\,.$$

As a result,

$$g(w,z) = \langle G(w,z), w - z\rangle.$$

We next normalize G and set

$$\eta(w,z) = \frac{G(w,z)}{g(w,z)}, \quad \text{for } w \in bD,\ z \in D.$$

Then η is a "generating form", namely $\langle \eta(w,z), w - z\rangle = 1$ for any $z \in D$ and for any w in a neighborhood of bD, see [LS-3, Lemma 6, Section 7]. Note that η is a form of type $(1,0)$ in w, with coefficients that are C^1 in w and C^∞ in z.

The Cauchy-Fantappié integral $\mathbf{C}^1$ is defined as

$$\mathbf{C}^1(f)(z) = \frac{1}{(2\pi i)^n}\int\limits_{w\in bD} f(w)\, j^*(\eta\wedge(\overline{\partial}\eta)^{n-1})(w,z), \quad z \in D.$$

Here f is an integrable function on bD and, as before, $j : bD \hookrightarrow \mathbb{C}^n$.

Note that

$$\overline{\partial}\eta = -\frac{1}{g^2}(\overline{\partial}g)\wedge G + \frac{1}{g}\,\overline{\partial}G = -\frac{1}{g}(\overline{\partial}g)\wedge\eta + \frac{1}{g}\,\overline{\partial}G,$$

and since $\eta \wedge \eta = 0$, it follows that

$$\mathbf{C}^1(f)(z) = \int\limits_{w \in bD} C^1(w,z) f(w), \quad z \in D, \tag{3.2}$$

where

$$C^1(w,z) = \frac{1}{(2\pi i)^n}\, j^*\left(\frac{G\wedge(\overline{\partial}G)^{n-1})(w,z)}{g(w,z)^n}\right).$$

Proposition 4. *Suppose that F is continuous in $\overline{D}$ and holomorphic in D. Let*

$$f = F\Big|_{bD}.$$

Then

$$\mathbf{C}^1(f)(z) = F(z), \quad z \in D.$$

Proof. This proposition is a restatement of [LS-3, Proposition 4, Section 7] (see also [LS-3, Proposition 2, Section 5] and [LS-3, Lemma 6, Section 7]). □

3.2. **Correction Operator.** While the Cauchy-Fantappié integral $\mathbf{C}^1$ reproduces holomorphic functions, $\mathbf{C}^1(f)$ is not holomorphic for general f. To achieve this we correct $\mathbf{C}^1$ by the solution of a $\overline{\partial}$-problem. Here we use the presentation of this idea as it appears in [LS-3, Section 8], where further details can be found; earlier versions are in [KS-2] and [Ra].

There is a C^∞-smooth, strongly pseudo-convex domain Ω that contains $\overline{D}$ with the property that

$$H(w,z) := \begin{cases} -\overline{\partial}_z\big(\eta\wedge(\overline{\partial}_w\eta)^{n-1}\big), & \text{for} \quad z \in \Omega \setminus \{|z-w| < \mu/2\} \\ \\ 0, & \text{for} \quad |z-w| < \mu/2 \end{cases}$$

is smooth in $z \in \Omega$ and continuous in $w \in bD$, and satisfies the compatibility condition $\overline{\partial}_z H(w,z) = 0$ whenever $z \in \Omega$ and $w \in bD$. So if we consider the solution of the $\overline{\partial}$-problem

$$\overline{\partial}_z C^2(w,z) = H(w,z), \quad z \in \Omega\,,$$

we can write $C^2(w,z) = \mathcal{S}_z(H(w,\cdot))$ for the corresponding normal solution operator $\mathcal{S}_z$, as given in e.g., [CS], [FK]. Then $C^2(w,z)$ is an $(n, n-1)$-form in w, whose coefficients are of class C^1 in w and depend smoothly on z. In particular, $C^2(w,z)$ is bounded on $bD \times \overline{D}$, and so

$$\sup_{(w,z)\in bD\times\overline{D}} |C^2(w,z)| \lesssim 1\,. \tag{3.3}$$

We then define

$$\mathbf{C}^2(f)(z) = \int_{bD} C^2(w,z) f(w) \tag{3.4}$$

and write

$$\mathbf{C}(f)(z) = \int_{bD} C(w,z) f(w)$$

where

$$C(w,z) = C^1(w,z) + C^2(w,z), \quad \text{and so} \quad \mathbf{C} = \mathbf{C}^1 + \mathbf{C}^2\,.$$

One has as a result

Proposition 5.

(1) *Whenever f is integrable, $\mathbf{C}(f)(z)$ is holomorphic for $z \in D$.*

(2) *If F is continuous in $\overline{D}$ and holomorphic in D and*

$$f = F\Big|_{bD}\,,$$

then $\mathbf{C}(f)(z) = F(z),\ z \in D.$

It is also useful to have the additional regularity of $\mathbf{C}(f)$ when f is Hölder in the sense of the quasi-distance $\mathtt{d}$, i.e. it satisfies

$$|f(w_1) - f(w_2)| \lesssim \mathtt{d}(w_1, w_2)^{\alpha} \quad \text{for some } 0 < \alpha \leq 1. \tag{3.5}$$

Proposition 6. *If f satisfies the Hölder-type condition* (3.5) *then* $\mathbf{C}(f)$ *extends to a continuous function on $\overline{D}$.*

Proof. Since $\mathbf{C} = \mathbf{C}^1 + \mathbf{C}^2$ and the kernel of $\mathbf{C}^2$ is continuous in $\overline{D}$, $\mathbf{C}^2(f)$ is automatically continuous there, and we are reduced to considering $\mathbf{C}^1(f)$.

Given the smoothness of $\eta(w,z)$ and $\overline{\partial}_w g(w,z)$ for $z \in D$, it suffices to prove the continuity of $\mathbf{C}^1(f)(z)$ for z in $\overline{D}$ and close to bD. To do this we set

$$F^{\delta}(z) = \mathbf{C}^1(f)(z^{\delta}), \quad \text{with } z^{\delta} = z + \delta\nu_z$$

with ν_z and δ as in Corollary 2. It will suffice to see that the functions F^{δ} converge uniformly on bD as $\delta \to 0$. In fact one can assert that

$$\sup_{z \in bD} |F^{\delta_1}(z) - F^{\delta_2}(z)| \lesssim \max\{\delta_1, \delta_2\}^{\alpha/2}. \tag{3.6}$$

This can be proved as follows. With $C^1(w,z)$ the kernel of the operator $\mathbf{C}^1$ we have

$$F^{\delta_1}(z)-F^{\delta_2}(z)=\int\limits_{w\in bD}\Big(C^1(w,z^{\delta_1})-C^1(w,z^{\delta_2})\Big)(f(w)-f(z)) \tag{3.7}$$

because $\mathbf{C}^1(1)=1$ by Proposition 4.

Now suppose $\delta_1\leq\delta_2$, then one has the following estimate

$$\left|C^1(w,z^{\delta_1})-C^1(w,z^{\delta_2})\right|\lesssim\min\left\{\frac{1}{\mathtt{d}(w,z)^{2n}},\ \frac{\delta_2}{\mathtt{d}(w,z)^{2n+2}}\right\}. \tag{3.8}$$

Indeed, $C^1(w,z)=j^*\left((2\pi i\,g(w,z))^{-2n}G\wedge(\overline{\partial}_wG)^{n-1}\right)$. Looking back at the definition of $G(w,z)$, see (3.1), we see that G and $\overline{\partial}_wG$ are bounded. So the inequality: $\left|C^1(w,z^{\delta_1})-C^1(w,z^{\delta_2})\right|\lesssim\mathtt{d}(w,z)^{-2n}$ follows from Corollary 2, if we take into account that $|g(w,z)|\approx\mathtt{d}(w,z)^2$.

Since ∇_zG and $\nabla_z\overline{\partial}_wG$ are also bounded, Corollary 2 again grants

$$\left|C^1(w,z^{\delta_1})-C^1(w,z^{\delta_2})\right|\lesssim|\delta_1-\delta_2|\,|g(w,z)|^{-n-1}\lesssim\delta_2\,\mathtt{d}(w,z)^{-2n-2},$$

since $\delta_2>\delta_1$. Hence the estimate (3.8) is established.

We now break the integration in (3.7) into two parts: where $\mathtt{d}(w,z)\leq\delta_2^{1/2}$, and $\mathtt{d}(w,z)>\delta_2^{1/2}$. Then since $|f(w_1)-f(w_2)|\lesssim\mathtt{d}(w,z)^{\alpha}$, the integral over the first part is bounded by a multiple of

$$\int\limits_{\mathtt{d}(w,z)\leq\delta_2^{1/2}}\mathtt{d}(w,z)^{-2n+\alpha}\,d\lambda(w)\,.$$

Similarly the integral over the second part is bounded by

$$\delta_2\int\limits_{\mathtt{d}(w,z)>\delta_2^{1/2}}\mathtt{d}(w,z)^{-2n-2+\alpha}\,d\lambda(w)\,.$$

Both integrals are $O(\delta_2^{\alpha/2})$ in view of (2.17) with $r=\delta_2^{1/2}$. Since we took $\delta_2\geq\delta_1$, this proves (3.6), and the proposition is established. □

4. A Cauchy transform; L^p-boundedness

Proposition 6 allows us to define the "Cauchy transform" $\mathcal{C}$. It is a linear operator, initially defined on functions satisfying (3.5) by

$$\mathcal{C}(f)=\mathbf{C}(f)\big|_{bD} \tag{4.1}$$

(We recall that $\mathbf{C}$ and thus $\mathcal{C}$ depend on a parameter ϵ, which here is kept fixed and hence omitted from the notations.)

It is worthwhile to point out the following formula for $\mathcal{C}(f)$

$$\mathcal{C}(f)(z) = f(z) \; + \int\limits_{w\in bD} C(w,z)\,[f(w)-f(z)]\,, \quad z\in bD\,, \tag{4.2}$$

which is proved by considering the identity

$$\mathbf{C}(f)(z^\delta) = f(z) \; + \int\limits_{w\in bD} C(w,z^\delta)\,[f(w)-f(z)]$$

where as before, $z^\delta = z+\delta\nu_z$, and letting $\delta\to 0$. (We remark that the integral in the expression above is absolutely convergent in view of the fact that f satisfies the Hölder-like condition (3.5).)

Our principal result for $\mathcal{C}$ is as follows.

Theorem 7. *The operator $\mathcal{C}$ initially defined for functions satisfying* (3.5) *extends to a bounded linear transformation on $L^p(bD,d\lambda)$, for $1<p<\infty$.*

On account of the equivalence (2.16) this also gives boundedness in $L^p(bD,d\sigma)$.

4.1. **Essential parts and remainders.** At several steps of our analysis we require a decomposition of the Cauchy integral $\mathbf{C}$ and the Cauchy transform $\mathcal{C}$ into an "essential part" plus an acceptable "remainder", and moreover such decomposition appears in several different forms. In fact, first we considered $\mathbf{C}=\mathbf{C}^1+\mathbf{C}^2$, with $\mathbf{C}^1$ a Cauchy-Fantappié integral and $\mathbf{C}^2$ a correction term via $\overline{\partial}$ (the "remainder"), and this led to the definition (4.1) of the Cauchy transform $\mathcal{C}$. A new decomposition of $\mathcal{C}$ is given immediately below; further decompositions of $\mathcal{C}$ will be needed in Part II, when we study the Cauchy-Szegő projection.

It is worthwhile to point out that the remainders that appear in all such decompositions will always be less singular than the corresponding essential parts, in the sense that their kernels (hereby generically denoted $R(w,z)$) are easily seen to be controlled via the improved bound

$$|R(w,z)| \lesssim \mathtt{d}(w,z)^{-2n+1} \tag{4.3}$$

for any $w,z\in bD$, and satisfy the improved difference condition

$$|R(w,z)-R(w,z')| \lesssim \frac{\mathtt{d}(z,z')}{\mathtt{d}(w,z)^{2n}} \tag{4.4}$$

whenever $\mathtt{d}(w,z)\geq c\,\mathtt{d}(z,z')$ for an appropriate large constant c.

By contrast, the kernels of the essential parts will exhibit no such improvements and will in fact retain the same singularities as the original transform $\mathcal{C}$, see for instance (4.18) below.

4.2. **A new decomposition of $\mathcal{C}$.** We begin by making a decomposition of the Cauchy transform $\mathcal{C}$ that will eventually allow us to study its "adjoint" on $L^2(bD, d\lambda)$. Here we take the numerator of the Cauchy-Fantappié integral, essentially $j^*\left(G \wedge (\overline{\partial} G)^{(n-1)}\right)/(2\pi i)^n$, see (3.2), and replace it by $j^*(\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1)})/(2\pi i)^n = d\lambda$, the Leray-Levi measure. So we define $\mathbf{C}^\sharp$ by

$$\mathbf{C}^\sharp(f) = \int\limits_{w\in bD} \frac{f(w)}{g(w,z)^n}\, d\lambda(w)\,, \quad z \in D. \tag{4.5}$$

Looking back at (3.1) we see that

$$G = \partial\rho + \gamma_0 + \sum_{k=0}^{n}(w_k - z_k)\gamma_k$$

where γ_0, γ_k are 1-forms whose coefficients are smooth in z and of class C^1 in w, with γ_k supported in $|w-z| \leq \mu$ and γ_0 supported where $|w-z| \geq \mu/2$. As a result

$$\mathbf{C}(f)(z) = \mathbf{C}^\sharp(f)(z) + \mathbf{R}(f)(z), \quad z \in D \tag{4.6}$$

where the remainder $\mathbf{R}$ is given by

$$\mathbf{R}f(z) = \int\limits_{bD} g(w,z)^{-n} j^*\big(\alpha_0(w,z) + \sum_j \alpha_j(w,z)(w_j - z_j)\big)\, f(w) \tag{4.7}$$
$$+ \quad \mathbf{C}^2(f)(z).$$

Here $\alpha_0(w,z)$ and $\alpha_j(w,z)$ are $(2n-1)$-forms in w whose coefficients are continuous in $w \in bD$ and smooth in $z \in \overline{D}$, with $\alpha_0(w,z)$supported away from the diagonal $\{w = z\}$. Also, $\mathbf{C}^2(f)$ is the correction term (3.4).

In view of (3.3) and of the fact that $|g(w,z)| \approx \mathtt{d}(w,z)^2$, we see that the kernel of $\mathbf{R}$ is bounded by a multiple of $\mathtt{d}(w,z)^{-2n+1}$, see (4.3). Thus, by (2.17) and Corollary 2 the integral defining $\mathbf{R}(f)$ converges absolutely and uniformly for $z \in D$ and hence extends to a continuous function on $\overline{D}$.

With this and (4.1) we can define the operator $\mathcal{C}^\sharp$ acting on f that satisfies (3.5) by

$$\mathcal{C}^\sharp(f) = \mathbf{C}^\sharp(f)\Big|_{bD}.$$

As a result we have a corresponding identity

$$\mathcal{C}(f)(z) = \mathcal{C}^\sharp(f)(z) + \mathcal{R}(f)(z), \quad z \in bD\,. \tag{4.8}$$

We point out that

$$\mathcal{C}^\sharp(f)(z) = \int\limits_{w\in bD} \frac{f(w)}{g(w,z)^n}\, d\lambda(w) \tag{4.9}$$

whenever f satisfies the Hölder-like condition (3.5) and $z \in bD$ is such that $f(z) = 0$. This results by a passage to the limit in (4.5) where we use (4.1) and (4.2). The remainder operator $\mathcal{R}$ will be dealt with in Section 4.5.

4.3. $\mathcal{C}^\sharp$ **as a "derivative".** A turning point in this analysis is the realization that $\mathcal{C}^\sharp(f)$ can be appropriately expressed in terms of df. For this purpose we define two operators $\mathbf{E}$ and $\mathbf{R}^\sharp$. Here the "essential part" $\mathbf{E}$ acts on continuous 1-forms μ on bD, and the output $\mathbf{E}(\mu)$ is the continuous function in $\overline{D}$ given by

$$\mathbf{E}(\mu)(z) = c_n \int\limits_{w\in bD} g(w,z)^{-n+1}\, \mu(w) \wedge j^*(\overline{\partial}\partial\rho(w,z))^{n-1}$$

with $c_n = 1/[(n-1)(2\pi i)^n]$. The "remainder" $\mathbf{R}^\sharp$ maps continuous functions on bD to continuous functions on $\overline{D}$ and is defined in a manner analogous to (4.7).

Proposition 8. *If $f \in C^1(bD)$, then*

$$\mathbf{C}^\sharp(f)(z) = \mathbf{E}(df)(z) + \mathbf{R}^\sharp(f)(z), \quad \textit{for } z \in D. \tag{4.10}$$

An analogous result can be found in [LS-4, Proposition 5.1]. As a consequence we have

$$\mathcal{C}^\sharp(f)(z) = \mathcal{E}(df)(z) + \mathcal{R}^\sharp(f)(z), \quad \text{for } z \in bD\,, \tag{4.11}$$

where $\mathcal{E}$ and $\mathcal{R}^\sharp$ are defined as the corresponding limits on bD of $\mathbf{E}$ and $\mathbf{R}^\sharp$.

The simple integration lemma below will be used here and at several later occasions. Suppose F and ρ are a pair of functions on bD with F of class C^1 and ρ of class C^2.

Lemma 9.

$$\int\limits_{bD} dF \wedge (\overline{\partial}\partial\rho)^{n-1} = 0\,.$$

If we make the stricter assumption that ρ is of class C^3, then since $d(\overline{\partial}\partial\rho) = 0$, the assertion of the lemma follows immediately from Stokes' theorem and the fact that $d(F \wedge (\overline{\partial}\partial\rho)^{n-1}) = dF \wedge (\overline{\partial}\partial\rho)^{n-1}$. The case when ρ is merely of class C^2 then follows this by approximating ρ in the C^2-norm by $\{\rho_k\}_k$ with each ρ_k of class C^3.

Proof of Proposition 8. We fix z in the interior of D; our goal is to apply Lemma 9 to

$$F(w) = f(w) \cdot g(w,z)^{-n+1}$$

(and ρ the defining function of our domain). We claim that computing $dF \wedge (\partial\overline{\partial}\rho)^{n-1}$ for such an F will give rise to several terms. One of these will be $-g(w,z)^{-n} f(w) d\lambda(w)$ (whose integral over bD is precisely $-\mathbf{C}^\sharp(f)(z)$), while the integrals of the other terms will be $\mathbf{E}(df)(z) + \mathbf{R}^\sharp(f)(z)$. Specifically, we have

$$dF = (-n+1) f(w) \cdot g(w,z)^{-n} d_w g(w,z) + df(w) \cdot g^{-n+1}(w,z),$$

and

$$d_w g = \partial\rho(w) + \beta_0(w,z) + \sum_j \beta_j(w,z)(w_j - z_j). \tag{4.12}$$

Here β_j and β_0 are 1-forms in w that are smooth as z varies over $\overline{D}$, and continuous as w varies over bD, with $\beta_0(w,z)$ supported away from the diagonal $\{w = z\}$. This leads to (4.10), thus proving the proposition. □

4.4. **The virtual adjoint of $\mathcal{C}^\sharp$.** Here it is crucial that we take the inner product, with respect to which the "adjoint" is defined, to be given by

$$(f_1, f_2) = \int_{bD} f_1(w)\, \overline{f_2}(w)\, d\lambda(w)\,,$$

with $d\lambda$ the Leray-Levi measure, and we will let $(\mathcal{C}^\sharp)^*$ denote the adjoint of $\mathcal{C}^\sharp$ with respect to such inner product. We have the following representation for $(\mathcal{C}^\sharp)^*$.

Proposition 10. *There is a linear operator $(\mathcal{C}^\sharp)^*$, acting on functions that are Hölder in the sense of* (3.5) *for some α, so that*

$$(\mathcal{C}^\sharp f_1, f_2) = (f_1, (\mathcal{C}^\sharp)^* f_2) \tag{4.13}$$

for any pair of such functions. Moreover

$$(\mathcal{C}^\sharp)^*(f)(z) = \int_{w\in bD} \overline{g}(z,w)^{-n} f(w)\, d\lambda(w) \tag{4.14}$$

for those $z \in bD$ such $f(z) = 0$.

(Notice that the integral above converges absolutely because when z is outside the support of f one has $|f(w)| \lesssim \mathtt{d}(w,z)^\alpha$.)

Proof. We consider the family of operators $\mathcal{C}^{\sharp,\delta}$, given by

$$\mathcal{C}^{\sharp,\delta}(f)(z) = \int\limits_{w\in bD} g(w, z^\delta)^{-n} f(w)\, d\lambda(w), \quad z \in bD\,,$$

where $z^\delta = z + \delta\nu_z$ is defined as in Corollary 2.

Now $g(w, z^\delta)^{-n}$ is uniformly bounded in z and w for each $\delta > 0$ (see Corollary 2), and hence for each δ the operator $\mathcal{C}^{\sharp,\delta}$ is bounded in $L^2(bD, d\lambda)$. Its (genuine) adjoint $(\mathcal{C}^{\sharp,\delta})^*$ is then given by

$$(\mathcal{C}^{\sharp,\delta})^*(f)(z) = \int\limits_{w\in bD} \overline{g}(z, w^\delta)^{-n} f(w)\, d\lambda(w) \tag{4.15}$$

where again $w^\delta = w + \delta\nu_w$, and $(\mathcal{C}^{\sharp,\delta}(f_1), f_2) = (f_1, (\mathcal{C}^{\sharp,\delta})^*(f_2))$, whenever $f_1,\ f_2 \in L^2(bD, d\lambda)$. It will suffice to see that whenever f is Hölder, $(\mathcal{C}^{\sharp,\delta})^*(f)$ converges uniformly, as $\delta \to 0$, to a limit, and to take $(\mathcal{C}^\sharp)^*(f)$ to be such limit. The convergence will be shown using Proposition 11 below, which is the special case when $f = 1$; we give a separate proof as its full conclusion will be needed later.

Proposition 11. *With same notations as above, we have*

$$(\mathcal{C}^{\sharp,\delta})^*(1) = \mathfrak{h}^*_\delta\,, \quad \delta > 0$$

where $\{\mathfrak{h}^*_\delta\}_\delta$ are continuous functions on bD. The $\mathfrak{h}^*_\delta$ converge uniformly to a continuous function $\mathfrak{h}^*$ as $\delta \to 0$.

Proof. The proof follows the spirit of Proposition 8, expressing in this case $(\mathcal{C}^{\sharp,\delta})^*(f)$ with $f = 1$ in terms of $df = 0$, with an acceptable error term. Here we will apply Lemma 9 with $F(w) = \overline{g}(z, w^\delta)^{-n-1}$.

If $|z - w| \le \mu/2$ then

$$d_w(\overline{g}(z, w^\delta)) =$$

$$= d_w\big(\langle\overline{\partial}\rho(z), \overline{z} - \overline{w}^\delta\rangle\big) - \frac{1}{2}\, d_w\left(\sum_{j,k} \overline{\tau}^{\,\epsilon}_{jk}(z)(\overline{z}_j - \overline{w}^\delta_j)(\overline{z}_k - \overline{w}^\delta_k)\right).$$

Now

$$d_w\big(\langle\overline{\partial}\rho(z), \overline{z} - \overline{w}\rangle\big) = -\overline{\partial}\rho(w) + \sum_j \left(\frac{\partial\rho(w)}{\partial\overline{w}_j} - \frac{\partial\rho(z)}{\partial\overline{z}_j}\right) d\overline{w}_j\,.$$

Also, $w^\delta = w + \delta\nu_w$, and if we write $\nu_w = (\nu_w^1, \ldots, \nu_w^n)$ then $d_w\nu_w$ are 1-forms with continuous coefficients. Altogether then

$$d_w\big(\overline{g}(z, w^\delta)\big) = -\overline{\partial}\rho(w) +$$

$$+\beta_0^* + \sum_j \beta_j^*(w,z)(\overline{w}_j - \overline{z}_j) + \sum_j \beta_j^{**}(w,z)\left(\frac{\partial\rho(w)}{\partial\overline{w}_j} - \frac{\partial\rho(z)}{\partial\overline{z}_j}\right) + \delta\,\gamma^\delta(z,w)\,.$$

Here β_0^*, β_j^*, β_j^{**} and γ^δ are 1-forms in w with coefficients that are C^1 in $z \in \overline{D}$ and in $w \in bD$; also $\gamma^\delta(z,w)$ is bounded as $\delta \to 0$, while $\beta_0^*(w,z)$ is supported away from the diagonal $\{w = z\}$.

Keeping in mind that $j^*d\rho = 0$ (or its equivalent formulation: $j^*\partial\rho = -j^*\overline{\partial}\rho$), we obtain from Lemma 9 that

$$\mathfrak{h}_\delta^* = (\mathcal{C}^{\sharp,\delta})^*(1) = \int\limits_{bD} \overline{g}(z, w^\delta)^{-n}\, d\lambda(w) \ = \ I_\delta^1 \ + \ I_\delta^2$$

with

$$I_\delta^1 = \int\limits_{bD} \overline{g}(z, w^\delta)^{-n} j^*\big[a_0(w,z) + \sum_{j=1}^n a_j(w,z)(\overline{w}_j - \overline{z}_j) + b_j(w,z)\big(\frac{\partial\rho}{\partial\overline{w}_j}(w) - \frac{\partial\rho}{\partial\overline{z}_j}(z)\big)\big]$$

where a_0, a_j and b_j are $(2n-1)$-forms in w, with coefficients that are continuous in $z \in D$ and $w \in bD$, with $a_0(w,z) = 0$ away from the diagonal $\{w = z\}$. From this it is evident that I_δ^1 converges uniformly as $\delta \to 0$ to a continuous function I^1.

Moreover

$$I_\delta^2 = \int\limits_{bD} j^*(A_\delta(w,z))$$

where $A_\delta(w,z)$ is a $(2n-1)$-form in w that satisfies the bound

$$|A_\delta(w,z)| \lesssim \delta\, |g_\epsilon(w,z)|^{-n}\,,$$

and it follows by (2.18) that $I_\delta^2 = O(\delta \log 1/\delta)$ as $\delta \to 0$, and hence this tends to zero uniformly in z, so that

$$\mathfrak{h}_\delta^*(z) \to I^1(z) =: \mathfrak{h}^*(z) \quad \text{uniformly in } \ z, \quad \text{as } \delta \to 0.$$

The proof of Proposition 11 is concluded. □

We may now complete the proof of Proposition 10 . Turning back to $(\mathcal{C}^{\sharp,\delta})^*(f)$ as expressed in (4.15) we see that

$$(\mathcal{C}^{\sharp,\delta})^*(f) = \int\limits_{bD} \overline{g}(z,w)^{-n}[f(w) - f(z)]\, d\lambda(w) + \mathfrak{h}_\delta^*(z) f(z)\,,$$

since $(\mathcal{C}^{\sharp,\delta})^*(1) = \mathfrak{h}^*_\delta$. Now if $|f(w) - f(z)| \lesssim \mathtt{d}(w,z)^\alpha$, $\alpha < 1$, we clearly have the convergence of $(\mathcal{C}^{\sharp,\delta})^*(f)$ as $\delta \to 0$, to a limit which we call $(\mathcal{C}^\sharp)^*(f)$ and which equals

$$(\mathcal{C}^\sharp)^*(f) = \int_{bD} \overline{g}(z,w)^{-n}[f(w) - f(z)]\, d\lambda(w) + \mathfrak{h}^*(z) f(z).$$

With this the duality identities (4.13) and (4.14) are established, proving Proposition 10. □

4.5. L^p**-boundedness of** $\mathcal{C}$**.** We shall prove the boundedness of $\mathcal{C}$ by applying to $\mathcal{C}^\sharp$ the $T(1)$ theorem in the form given in [C-1]. To this end, in this section we go over the key hypotheses of this theorem, namely

- the cancellation properties of $\mathcal{C}^\sharp$;
- the action of $\mathcal{C}^\sharp$ on "bump functions";
- the difference estimates for the kernel of $\mathcal{C}^\sharp$.

We first recall the cancellation properties of $\mathcal{C}^\sharp$. We have

$$\mathcal{C}^\sharp(1) = \mathfrak{h} \quad \text{and} \quad (\mathcal{C}^\sharp)^*(1) = \mathfrak{h}^*$$

where both $\mathfrak{h}$ and $\mathfrak{h}^*$ are continuous functions. This is what was proved for $(\mathcal{C}^\sharp)^*(1)$ in Proposition 11. The proof works as well for $\mathcal{C}^\sharp(1)$, using (4.11), by which $\mathcal{C}^\sharp(1) = \mathcal{R}^\sharp(1)$. Now $\mathcal{R}^\sharp(1)$ can be treated like I^1_δ in the proof of Proposition 11.

Next we come to the action of $\mathcal{C}^\sharp$ on "bump functions". We fix α, $0 < \alpha < 1$, and say that f is a *normalized bump function associated with a boundary ball* $\mathbf{B}_r(\hat{w})$ (with $\hat{w} \in bD$), if it is supported in $\mathbf{B}_r(\hat{w})$ and satisfies

$$|f(z)| \leq 1, \quad |f(z) - f(z')| \leq \left(\frac{\mathtt{d}(z,z')}{r}\right)^\alpha, \quad \text{for all} \quad z, z' \in bD.$$

The main fact we use for these bump functions is a consequence of Lemma 12 below.

Suppose T is a linear transformation defined on functions satisfying the Hölder-type regularity (3.5) for some $\alpha > 0$, and mapping these to continuous functions on bD. Assume further that T has a kernel $K(w,z)$ so that

$$T(h)(z) = \int_{bD} K(w,z) h(w)\, d\lambda(w) \tag{4.16}$$

holds whenever $h(z) = 0$, with $|K(w,z)| \leq \mathtt{d}(w,z)^{-2n}$. Suppose in addition that

$$|T(f_0)(\hat{w})| \lesssim 1 \tag{4.17}$$

whenever f_0 is a C^1-smooth function supported in a ball $\mathbf{B}_r(\hat{w})$, and $|f_0| \leq 1$ and $|\nabla f_0| \leq 1/r^2$ on bD.

Lemma 12. *With the above assumptions, the following holds whenever f is a normalized bump function associated to $\mathbf{B}_r(\hat{w})$:*

(a) $\sup_{z\in bD} |T(f)(z)| \lesssim 1,$ *and*

(b) $\|T(f)\|^2_{L^2(bD,d\lambda)} \lesssim r^{2n}.$

Proof. Let f be a given normalized bump function associated with $\mathbf{B}_r(\hat{w})$, and let χ be a non-negative C^1-smooth function on $\mathbb{C}$, so that $\chi(u+iv) = 1$ for $|u+iv| \leq 1/2$, $\chi(u+iv) = 0$ for $|u+iv| \geq 1$ and, furthermore, $|\nabla\chi(u+iv)| \leq 1/r^2$. Set

$$\widetilde{\chi}_{r,\hat{w}}(w) = \chi\left(\frac{\mathrm{Im}\langle\partial\rho(\hat{w}), \hat{w}-w\rangle}{c\,r^2} + i\,\frac{|\hat{w}-w|^2}{c\,r^2}\right) \quad \text{for any} \quad w \in bD.$$

If the constant c is chosen sufficiently large, it follows from (2.9) that $\widetilde{\chi}_{r,\hat{w}}(w) = 0$ if $\mathtt{d}(\hat{w}, w) \geq r$, while $\widetilde{\chi}_{r,\hat{w}}(w) = 1$ whenever $\mathtt{d}(\hat{w}, w) \leq c'r$ (with c' another constant). Now define

$$f_0(w) = f(\hat{w})\,\widetilde{\chi}_{r,\hat{w}}(w).$$

It is clear that f_0 satisfies all the requirements ensuring that (4.17) holds. We make the following four assertions:

(1) $|(f - f_0)(w)| \lesssim \left(\frac{\mathtt{d}(w,\hat{w})}{r}\right)^{\alpha};$

(2) $|T(f)(\hat{w})| \lesssim 1;$

(3) $|T(f)(z)| \lesssim 1$ for any $z \in \mathbf{B}_{cr}(\hat{w})$;

(4) $|T(f)(z)| \lesssim \mathtt{d}(z,\hat{w})^{-2n} r^{2n}$ for any $z \notin \mathbf{B}_{cr}(\hat{w})$.

Note that conclusion *(a)* would then follow at once from assertions *(3)* and *(4)*; to prove conclusion *(b)* we first write

$$\|T(f)\|^2_{L^2(bD)} = \int\limits_{\mathbf{B}_{cr}(\hat{w})} |T(f)|^2 \quad + \int\limits_{bD\setminus\mathbf{B}_{cr}(\hat{w})} |T(f)|^2 \quad = \quad I + II,$$

and observe that $I \lesssim r^{2n}$ by conclusion *(a)* and similarly, $II \lesssim r^{2n}$ by assertion *(4)* and (2.17) (with $\beta = 2n$).

It remains to prove the four assertions. To prove the first assertion, note that since $f_0(\hat{w}) = f(\hat{w})$, we may write $f(w) - f_0(w) = I + II$,

where $I = f(w) - f(\hat{w})$, $II = f_0(\hat{w}) - f_0(w)$, and in view of our hypotheses on f, we only need to show that

$$|II| = |f_0(\hat{w}) - f_0(w)| \lesssim \left(\frac{\mathtt{d}(w,\hat{w})}{r}\right)^{\alpha}.$$

To this end note that

$$f_0(w) - f_0(\hat{w}) = f(\hat{w})\left(\widetilde{\chi}_{r,\hat{w}}(w) - \widetilde{\chi}_{r,\hat{w}}(\hat{w})\right)$$

(because $\widetilde{\chi}_{r,\hat{w}}(\hat{w}) = 1$). By the mean value theorem we have that the right-hand side of this identity is equal to

$$O\left(\frac{|\mathrm{Im}\langle\partial\rho(\hat{w}), \hat{w} - w\rangle| + |\hat{w} - w|^2}{c\,r^2}\right)$$

and by (2.9) and (2.12) this is bounded by

$$\left(\frac{\mathtt{d}(w,\hat{w})}{r}\right)^{2} \leq \left(\frac{\mathtt{d}(w,\hat{w})}{r}\right)^{\alpha},$$

(recall that $\alpha < 1$) thus proving assertion *(1)*. To prove the second assertion, we write

$$T(f) = T(f_0) + T(f - f_0).$$

Now we have that $|T(f_0)(\hat{w})| \lesssim 1$ by (4.17), and $(f - f_0)(\hat{w}) = 0$ by the definition of f_0. It follows that

$$|T(f - f_0)(\hat{w})| = \left|\int_{bD} K(w,\hat{w})\,(f - f_0)(w)\,d\lambda(w)\right| \lesssim \frac{1}{r^{\alpha}}\int_{\mathbf{B}_r(\hat{w})} \mathtt{d}(w,\hat{w})^{-2n+\alpha} d\lambda(w)$$

by our hypotheses on T and K along with assertion *(1)*; the inequality $|T(f - f_0)| \lesssim 1$ now follows by combining the above with (2.17). To prove assertion *(3)* we think of f as (a multiple of) a bump function associated with a ball $\mathbf{B}_{c'r}(z)$ for a suitable constant c', and we apply assertion *(2)*. We are left to prove assertion *(4)*. To this end, we first note that $f(z) = 0$ because $z \notin \mathbf{B}_{cr}(\hat{w})$, so by our hypotheses on T and $K(w,z)$ we have that

$$|Tf(z)| \lesssim \int_{\mathbf{B}_r(\hat{w})} \mathtt{d}(w,z)^{-2n}\,d\lambda(w)$$

(recall that $|f| \leq 1$). Now it follows by the triangle inequality that

$$\mathtt{d}(w,z) \gtrsim \mathtt{d}(z,\hat{w}) \quad \text{whenever} \quad z \notin \mathbf{B}_{cr}(\hat{w}), \quad w \in \mathbf{B}_r(\hat{w}),$$

and c is sufficiently large, thus proving assertion *(4)*. $\square$

Proposition 13. *If $\hat{w} \in bD$ and f is a normalized bump function associated with $\mathbf{B}_r(\hat{w})$, then*

$$\sup_{z\in bD} |\mathcal{C}^\sharp(f)(z)| \lesssim 1$$

and

$$\|\mathcal{C}^\sharp(f)\|_{L^2(bD,d\lambda)} \lesssim r^n.$$

Proof. The proof is an application of Lemma 12 to $T = \mathcal{C}^\sharp$. Then the representation for T, (4.16), follows from (4.9), while the estimate for the kernel $K(w,z)$ of T as above is an immediate consequence of the definition, see (2.12). It remains to check that the estimate $|T(f_0)(\hat{w})| \lesssim 1$ holds whenever f_0 is a C^1-smooth function supported in a ball $\mathbf{B}_r(\hat{w})$, with $|f_0| \leq 1$ and $|\nabla f_0| \leq 1/r^2$ on bD: but this is a consequence of (4.6) and the key identity (4.11) and, in addition, the fact that the kernels of the operators $\mathcal{R}$ and $\mathcal{R}^\sharp$, occurring in (4.8) and (4.11), are each bounded by $c\,\mathtt{d}(\hat{w},w)^{-2n+1}$, see (4.3), while the kernel of the operator $\mathcal{E}$ is bounded by $c\,\mathtt{d}(\hat{w},w)^{-2n+2}$, see (4.4) and (2.17). □

Remark A. The first conclusion in Proposition 13 also holds for $\mathcal{C}$ because the difference $(\mathcal{C}^\sharp - \mathcal{C})(f)$ is easily seen to be bounded for f bounded.

The last point about $\mathcal{C}^\sharp$ that is needed are the usual difference estimates for its kernel $g(w,z)^{-n}$. Besides the inequality $|g(w,z)|^{-n} \lesssim \mathtt{d}(w,z)^{-2n}$ we have

$$|g(w,z)^{-n} - g(w,z')^{-n}| \lesssim \frac{\mathtt{d}(z,z')}{\mathtt{d}(w,z)^{2n+1}} \tag{4.18}$$

and

$$|g(w,z)^{-n} - g(w',z)^{-n}| \leq c_\epsilon\, \frac{\mathtt{d}(w,w')}{\mathtt{d}(w,z)^{2n+1}} \tag{4.19}$$

whenever $\mathtt{d}(w,z) \geq c\,\mathtt{d}(z,z')$ for an appropriate large constant c. In fact when $\mathtt{d}(w,z) \geq c\,\mathtt{d}(z,z')$ for large c, one has that $|g(w,z)| \approx |g(w,z')|$, and so

$$|g(w,z)^{-n} - g(w,z')^{-n}| \lesssim \frac{|g(w,z) - g(w,z')|}{|g(w,z)|^{n+1}}\,.$$

Thus, to prove (4.18) we need only to invoke (2.15) (applied to $A(w,z) = g_\epsilon(z,w)$, however with the roles of w and z interchanged with one another) and the fact that $|g(w,z)| \approx \mathtt{d}(w,z)^2$. The inequality (4.19) is proved similarly.

We may now set

$$T = \mathcal{C}^\sharp,$$

and apply the $T(1)$-theorem to establish the $L^p(bD, d\lambda)$-boundedness of such T. In the terminology of [C-2, Chapter IV] we identify the space X with bD, points x, y in X with points z, w in bD, the quasi-distance ρ with $\mathtt{d}$, the measure μ with λ, and the kernel K with $g(w, z)^{-n}$.

Then by what we have just shown $T(1)$ and $T^*(1)$ are continuious functions on bD. Also $|(Tf_1, f_2)| \lesssim r_1^n r_2^n$, whenever f_1 and f_2 are normalized bump functions associated to boundary balls $\mathbf{B}_{r_1}(\hat{w}_1)$ and $\mathbf{B}_{r_2}(\hat{w}_2)$, respectively. This follows immediately from Proposition 13. In view of (4.9), the kernel $K(w, z) = g(w, z)^{-n}$ has the property that

$$(Tf, h) = \int\limits_{bD \times bD} K(w, z)\, f(w)\, \overline{h}(z)\, d\lambda(w)\, d\lambda(z)$$

whenever the functions f and h have disjoint support and are Hölder in the sense of (3.5). Also, $K(w, z)$ satisfies the difference conditions

$$|K(w, z) - K(w, z')| \lesssim \frac{\mathtt{d}(z, z')^\alpha}{\mathtt{d}(w, z)^{2n+\alpha}} \quad \text{when} \quad \mathtt{d}(w, z) \geq c\, \mathtt{d}(z, z')$$

and

$$|K(w, z) - K(w', z)| \lesssim \frac{\mathtt{d}(w, w')^\alpha}{\mathtt{d}(w, z)^{2n+\alpha}} \quad \text{when} \quad \mathtt{d}(w, z) \geq c\, \mathtt{d}(w, w').$$

This is because $g(w, z)^{-n}$ satisfies these properties for $\alpha = 1$ by (4.18) and (4.19), and hence for $\alpha < 1$ and in particular for

$$\alpha > 0. \tag{4.20}$$

From these, [C-2, Theorem 13] guarantees that T extends to a bounded linear operator on $L^p(bD, d\lambda)$, for each $1 < p < \infty$. However

$$\mathcal{C} \;=\; T + \mathcal{R}$$

by (4.8), since $T = \mathcal{C}^\sharp$. Next recall that if $R(w, z)$ is the kernel of $\mathcal{R}$, it follows by the description of $\mathcal{R}$ (essentially given in (4.7)) that $|R(w, z)| \lesssim \mathtt{d}(w, z)^{-2n+1}$, see (4.3), and as a result

$$\sup_{z \in bD} \int\limits_{w \in bD} |R(w, z)|\, d\lambda(w) < \infty$$

and

$$\sup_{w \in bD} \int\limits_{z \in bD} |R(w, z)|\, d\lambda(z) < \infty\,.$$

Now it is well known that if $\mathcal{T}$ is an operator whose kernel $\mathcal{T}(w,z)$ satisfies

$$\sup_{z\in bD}\int\limits_{w\in bD}|\mathcal{T}(w,z)|\,d\lambda(w)\leq 1\,,\ \text{and}\ \sup_{w\in bD}\int\limits_{z\in bD}|\mathcal{T}(w,z)|\,d\lambda(z)\leq 1 \tag{4.21}$$

then

$$\|\mathcal{T}\|_{L^p\to L^p}\leq 1\,. \tag{4.22}$$

Hence Theorem 7 is proved if we take $\mathcal{T}=c\,\mathcal{R}$.

4.6. **Further regularity.** It will be useful to have the following regularity property of $\mathcal{C}$.

Proposition 14. *For any $0<\alpha<1$, the transform $\mathcal{C}: f\mapsto \mathcal{C}(f)$ preserves the space of Hölder-like functions satisfying condition* (3.5).

Proof. The proof of this result follows the same lines as [LS-4, Proposition 6.3] and therefore we shall be brief. Fix z_1 and z_2 in bD, and consider the boundary ball $\mathbf{B}_r(z_1)=\{w\in bD\ :\ \mathtt{d}(z_1,w)<r\}$ with radius $r=\mathtt{d}(z_1,z_2)$, and let

$$\widetilde{\chi}_{r,z_1}(w)=\chi\left(\frac{\operatorname{Im}\langle\partial\rho(w),w-z_1\rangle}{c\,r^2}+i\frac{|w-z_1|^2}{c\,r^2}\right)$$

be the special cutoff function, supported in this ball, that was constructed in the proof of Lemma 12 (with the center now at z_1). At this stage one invokes (4.2) for $z=z_j$, $j=1,2$, and thus one writes $\mathcal{C}(f)(z_j)$ as follows:

$$\mathcal{C}(f)(z_j)=I_j+II_j+f(z_j),\quad j=1,2$$

where

$$I_j=\int\limits_{w\in bD}C(w,z_j)\widetilde{\chi}_{r,z_1}(w)(f(w)-f(z_j))$$

and

$$II_j=\int\limits_{w\in bD}C(w,z_j)(1-\widetilde{\chi}_{r,z_1}(w))(f(w)-f(z_j))$$

with $C(w,z)$ the kernel of $\mathcal{C}$. The first observation is then that each of $|I_1|$ and $|I_2|$ is majorized by a constant multiple of $\mathtt{d}(z_1,z_2)^\alpha$ (this is because the integrands are majorized by $\mathtt{d}(w,z_j)^{-2n+\alpha}$, and then one uses (2.17) with $\beta=\alpha$.) Next one shows that $|II_1-II_2|$ is also

majorized by a constant multiple of $\mathtt{d}(z_1, z_2)^\alpha$. To see this, one further decomposes the term II_2 as follows

$$II_2 = \widetilde{II}_2 + (f(z_1) - f(z_2)) \int\limits_{w \in bD} C(w, z_2)(1 - \widetilde{\chi}_{r,z_1}(w))$$

with

$$\widetilde{II}_2 = \int\limits_{w \in bD} C(w, z_2)(1 - \widetilde{\chi}_{r,z_1}(w))(f(w) - f(z_1))\,.$$

Then one observes that the difference $|II_1 - \widetilde{II}_2|$ is majorized by $\mathtt{d}(z_1, z_2)^\alpha$ by invoking the decomposition $\mathcal{C} = \mathcal{C}^\sharp + \mathcal{R}$ and then using the estimate (4.18) for $|g^{-n}(w, z_1) - g^{-n}(w, z_2)|$ (the difference estimate for the kernel of $\mathcal{C}^\sharp$) and the similar but easier estimate (4.4) for the remainder operator $\mathcal{R}$, along with the integral estimate (2.17) with $\beta = 1$. Finally, the integral in the remaining term in II_2, $\int C(w, z_2)(1 - \widetilde{\chi}_{r,z_1}(w))$, is uniformly bounded, because $\mathcal{C}(1) = 1$ and $\mathcal{C}(\widetilde{\chi}_{r,z_1})$ is uniformly bounded by Remark A, since $\widetilde{\chi}_{r,z_1}$ is a bump function. The proof of Proposition 14 is concluded. □

Remark B. The proof of Proposition 14 also shows that $\mathcal{C}^\sharp : f \mapsto \mathcal{C}^\sharp(f)$ preserves the space of Hölder-like functions satisfying condition (3.5). So in particular $\mathfrak{h} = \mathcal{C}^\sharp(1)$ is Hölder-continuous of order α in the sense of (3.5) for any $0 < \alpha < 1$, and one could in fact prove that the same is true for $\mathfrak{h}^* = (\mathcal{C}^\sharp)^*(1)$; see [LS-4, Corollary 4 and (5.16)] for a similar result. These improved cancellation conditions would then allow to reduce the application of the $T(1)$-theorem for $\mathcal{C}^\sharp$ (and therefore $\mathcal{C}$) to the simpler situation when $T(1) = 0 = T^*(1)$, as was done in [LS-4, Section 6.3] for the Cauchy-Leray integral of a strongly $\mathbb{C}$-linearly convex domain. However in the present context we will not pursue this approach, as a further application of the $T(1)$-theorem (for a related operator) will be needed when we deal with the Cauchy-Szegő projection in Part II below, but in that context the cancellation conditions for $T(1)$ and $T^*(1)$ cannot be improved beyond continuity on bD.

Part II: The Cauchy-Szegő projection

We now come to the main result of this paper: the $L^p(bD)$-regularity of the Cauchy-Szegő projection for $1 < p < \infty$ (Theorem 15).

As mentioned earlier, in defining the Cauchy-Szegő projection it is imperative to specify the underlying measure for bD that arises in the notion of orthogonality that is used. To put the matter precisely, suppose $d\sigma$ is the induced Lebesgue measure on bD, and consider also

a weight function ω which is strictly positive and continuous on bD. Then the Lebesgue spaces, $L^p(bD, d\sigma)$ and $L^p(bD, \omega\, d\sigma)$ (with norms $\|f\|_{L^p(bD,\, d\sigma)} = (\int |f|^p d\sigma)^{1/p}$ and $\|f\|_{L^p(bD,\, \omega\, d\sigma)} = (\int |f|^p \omega\, d\sigma)^{1/p}$ respectively) are *equivalent* in the sense that *1.* the two spaces consist of the same elements, and *2.* the two norms are comparable, that is

$$\|f\|_{L^p(bD,\, d\sigma)} \approx \|f\|_{L^p(bD,\, \omega\, d\sigma)}.$$

So in this way we can speak about the boundedness on $L^p(bD)$ of the Cauchy transform $\mathcal{C}$ we studied in Part I, without here specifying the particular strictly positive continuous weight function used. However the measures $d\sigma$ and $\omega\, d\sigma$ give rise to different inner products on $L^2(bD)$, that are respectively

$$(f_1, f_2)_1 = \int\limits_{bD} f_1 \overline{f_2}\, d\sigma \quad \text{and} \quad (f_1, f_2)_\omega = \int\limits_{bD} f_1 \overline{f_2}\, \omega\, d\sigma\,.$$

With this in mind, we define the Cauchy-Szegő projection $\mathcal{S}_\omega$ to be the orthogonal projection of $L^2(bD, \omega\, d\sigma)$, where orthogonality is taken with respect to the inner product $(\cdot,\cdot)_\omega$, onto the *Hardy Space* $\mathcal{H}^2(bD, \omega\, d\sigma)$, which we presently define as the closure in $L^2(bD, \omega\, d\sigma)$ of the set of boundary values of those functions that are continuous on $\overline{D}$ and holomorphic in D. (Further characterizations and representations of $\mathcal{H}^2(bD, \omega\, d\sigma)$ and more generally, of $\mathcal{H}^p(bD, \omega\, d\sigma)$, are given in [LS-5].) In view of the above, the usual Cauchy-Szegő projection is $\mathcal{S}_1$, and the Cauchy-Szegő projection with respect to the Leray-Levi measure $d\lambda$ is then $\mathcal{S}_\lambda$, where $d\lambda = \Lambda\, d\sigma$, see (2.16). While there is no simple and direct link connecting any two of these projections, the Cauchy-Szegő projection $\mathcal{S}_\lambda$ with respect to the Leray-Levi measure provides the way to understanding the projection $\mathcal{S}_1$ (and all the others, the totality of $\mathcal{S}_\omega$). For this reason we study $\mathcal{S}_\lambda$ first.

5. The Cauchy-Szegő projection: case of the Leray-Levi measure

5.1. **Statement of the main results.** In this section it will be convenient to simplify the notation, dropping the subscript λ, and to denote $\mathcal{S}_\lambda$ by $\mathcal{S}$, $L^p(bD, d\lambda)$ by $L^p(bD)$, and so forth. Our main results are as follows.

Theorem 15 (Main Theorem)**.** *The operator $\mathcal{S}$, initially defined on $L^2(bD)$, extends to a bounded operator on $L^p(bD)$, for $1 < p < \infty$.*

In the next section we prove the general result

Theorem 16. *The same conclusion holds for $\mathcal{S}_\omega$, whenever ω is a strictly positive and continuous function on bD.*

5.2. **Outline of the proof of Theorem 15.** In what follows it will be important to keep track of the dependence of the Cauchy transform on ϵ, so we will revert to writing the Cauchy transform as $\mathcal{C}_\epsilon$, with $\mathcal{C}_\epsilon$ equaling the $\mathcal{C}$ that appears in Section 4. Similarly we will write $\mathcal{R}^\sharp_\epsilon$ for $\mathcal{R}^\sharp$, g_ϵ for g, etc.

As we pointed out in the introduction, one of the main thrusts in the proof of Theorem 15 is a comparison of the Cauchy-Szegő projection with the Cauchy transforms $\{\mathcal{C}_\epsilon\}$ that were discussed in Part I. Such comparison is effected in Proposition 17 below, whose proof is given in [LS-5].

Proposition 17. *As operators on $L^2(bD)$ we have*

(a) $\mathcal{C}_\epsilon \mathcal{S} = \mathcal{S}$

(b) $\mathcal{S}\mathcal{C}_\epsilon = \mathcal{C}_\epsilon$

This proposition immediately implies

(c) $\mathcal{S}(I + \mathcal{C}_\epsilon - \mathcal{C}_\epsilon^*) = \mathcal{C}_\epsilon\,,$

as can be seen by taking adjoints of identity *(a)* which gives $\mathcal{S}\mathcal{C}_\epsilon^* = \mathcal{S}$, and subtracting this from *(b)*. Here the super-script * denotes the adjoint with respect to the inner product

$$(f_1, f_2)_\lambda = \int\limits_{bD} f_1 \overline{f_2}\, d\lambda\,,$$

for which $\mathcal{S}^* = \mathcal{S}$. An identity analogous to *(c)* was used in [KS-2] to study the Cauchy-Szegő projection in the situation when D is smooth (and strongly pseudo-convex). In that situation there was only the case $\epsilon = 0$ and the analogue of $\mathcal{C}_\epsilon - \mathcal{C}_\epsilon^*$ was "small" (in fact smoothing), and $I + \mathcal{C}_\epsilon - \mathcal{C}_\epsilon^*$ was "inverted" by a partial Neumann series. By contrast, when D is of class C^2 (as opposed to smooth) we will see below that $\|\mathcal{C}_\epsilon\|_{L^p \to L^p}$ may tend to ∞ as $\epsilon \to 0$ and, similarly, that no appropriate control on the size of $\mathcal{C}_\epsilon - \mathcal{C}_\epsilon^*$ can be expected as $\epsilon \to 0$. What works instead is to truncate the transform $\mathcal{C}_\epsilon$. We shall write

$$\mathcal{C}_\epsilon = \mathcal{C}_\epsilon^s + \mathcal{R}_\epsilon^s$$

where the kernel of $\mathcal{C}_\epsilon^s$ agrees with that of $\mathcal{C}_\epsilon$ when $\mathtt{d}(w,z) \lesssim s$ and vanishes when $\mathtt{d}(w,z) \gtrsim s$. If this cutoff is done appropriately, we then have the following key fact.

Proposition 18. *For any $\epsilon > 0$ there is an $s(\epsilon) > 0$, so that when $s \leq s(\epsilon)$*

$$\|\mathcal{C}^s_\epsilon - (\mathcal{C}^s_\epsilon)^*\|_{L^p \to L^p} \lesssim \epsilon^{1/2} M_p, \quad 1 < p < \infty\,. \tag{5.1}$$

Here the bound M_p is independent of ϵ and depends on p as follows:

$$M_p = \frac{p}{p-1} + p.$$

Remark C. The proof also shows that $\epsilon^{1/2}$ can be replaced by ϵ^β for any $0 < \beta < 1$ (but not $\beta = 0$, nor $\beta = 1$). However any bound that tends to zero with ϵ will suffice in our application below.

Proof of Theorem 15. Let us see how Proposition 18 proves Theorem 15. Consider the case $1 < p \leq 2$. Since $\mathcal{C}_\epsilon = \mathcal{C}^s_\epsilon + \mathcal{R}^s_\epsilon$ we have by Proposition 17 (in fact identity *(c)*) that

$$\mathcal{C}_\epsilon + \mathcal{S}(\mathcal{R}^s_\epsilon)^* - \mathcal{S}\mathcal{R}^s_\epsilon = \mathcal{S}\big(I + \mathcal{C}^s_\epsilon - (\mathcal{C}^s_\epsilon)^*\big). \tag{5.2}$$

However $I + \mathcal{C}^s_\epsilon - (\mathcal{C}^s_\epsilon)^*$ is invertible as a bounded operator on $L^p(bD)$ when ϵ and s are taken sufficiently small, by Proposition 18, as can be seen by applying a Neumann series when $\epsilon^{1/2} M_p << 1$.

Next, the kernel of $\mathcal{R}^s_\epsilon$ is supported where $\mathtt{d}(w,z) \gtrsim s$, and can be seen to be bounded by $c'_\epsilon s^{-1}\mathtt{d}(w,z)^{-2n+1}$, see (4.3). Thus $\mathcal{R}^s_\epsilon$ maps $L^1(bD)$ to $L^\infty(bD)$ (but the norm $\|\mathcal{R}^s_\epsilon\|_{L^1 \to L^\infty}$ is not bounded as ϵ and s tend to zero!). The same may be said of $(\mathcal{R}^s_\epsilon)^*$.

So $\mathcal{R}^s_\epsilon$ and $(\mathcal{R}^s_\epsilon)^*$ map $L^p(bD)$ to $L^2(bD)$, while $\mathcal{S}$ maps $L^2(bD)$ to $L^2(bD)$, and hence it maps $L^2(bD)$ to $L^p(bD)$, because $p \leq 2$. (Here we use the hypotesis that D, and hence bD, is bounded.)

Altogether then the left-hand side of (5.2) is bounded on $L^p(bD)$, in view of the corresponding boundedness of $\mathcal{C}_\epsilon$, Theorem 7. Applying $(I + (\mathcal{C}^s_\epsilon - (\mathcal{C}^s_\epsilon)^*)^{-1}$ to both sides of (5.2) yields the boundedness of $\mathcal{S}$ on $L^p(bD)$, when $1 < p \leq 2$. The case $p > 2$ follows by duality. □

At this point we digress briefly to state, for the sake of comparison, a companion result that will be needed in Section 6, when we will deal with the Cauchy-Szegő projection for $L^2(bD, d\sigma)$.

Proposition 19. *For any $\epsilon > 0$ there is an $s(\epsilon) > 0$, so that when $s \leq s(\epsilon)$*

$$\|\,\mathcal{C}^s_\epsilon\,\|_{L^p \to L^p} \lesssim M_p\,, \quad \textit{for } 1 < p < \infty. \tag{5.3}$$

Note that as opposed to (5.1) in Proposition 18, here we cannot have a gain in ϵ by making s small.

5.3. **Proof of Proposition 18.** As stated above, the norm of the Cauchy transform $\mathcal{C}_\epsilon$ may be unbounded when $\epsilon \to 0$. This is due to the fact that we have replaced the continuous functions $\partial^2\rho(w)/\partial w_j \partial w_k$ by their C^1 approximations $\tau^\epsilon_{j,k}(w)$, and the quantities

$$c_\epsilon = \sup_{\substack{w\in bD \\ 1\le j,k\le n}} |\nabla \tau^\epsilon_{j,k}(w)| \tag{5.4}$$

which first occurred in (2.14) and appear again below, will in general tend to infinity as $\epsilon \to 0$ in a manner that reflects the modulus of continuity of the $\partial^2\rho(w)/\partial w_j \partial w_k$. The constants c_ϵ first appeared in (2.14); they also occurred implicitly at various other stages when first derivatives of the $\tau^\epsilon_{j,k}(w)$ were involved. Among these instances are the definition of the Cauchy integral (3.2) via the form $\overline{\partial} G$ (see (3.1)), and the coefficients $\alpha_j(w,z)$ that enter in the formula (4.7). These facts are all working against us as we try to control the growth of $\mathcal{C}_\epsilon$ (or $\mathcal{C}^s_\epsilon$). However, not hindering us is the simple observation that was used before, see (2.4), namely the fact that

$$|g_\epsilon(w,z)| \approx |g_0(w,z)| = \mathtt{d}(w,z)^2\,,$$

with the implied bounds not dependent on ϵ.

Moreover, what really helps us are two lemmas below. The first shows that under the right circumstances we can remove the bound c_ϵ from (2.14).

Lemma 20. *For every $\epsilon > 0$ there is an $s = s(\epsilon)$, so that if $s \le s(\epsilon)$ and $\mathtt{d}(w,z) \le s$, $\mathtt{d}(w',z) \le s$, then*

$$|g_\epsilon(w,z) - g_\epsilon(w',z)| \lesssim \mathtt{d}(w,w')^2 + \mathtt{d}(w,w')\,\mathtt{d}(w,z)\,. \tag{5.5}$$

Indeed, recalling the proof of (2.14) we see that it gives

$$|g_\epsilon(w,z) - g_\epsilon(w',z)| \lesssim \mathtt{d}(w,w')^2 + \mathtt{d}(w,w')\,\mathtt{d}(w,z) + |\,II\,|\,,$$

with $II = Q_w(w'-z) - Q_{w'}(w'-z)$, and in (2.15) it was proved that

$$|\,II\,| \lesssim c_\epsilon\,|w-w'|\,|w'-z|^2$$

where c_ϵ is as in (5.4). Also $|w-w'| \lesssim \mathtt{d}(w,w')$, $|w'-z| \lesssim \mathtt{d}(w',z)$, and $\mathtt{d}(w',z) \lesssim \mathtt{d}(w,w') + \mathtt{d}(w,z)$ by the triangle inequality.

Thus if we take $s_0(\epsilon) = c_\epsilon^{-1}$, then

$$|\,II\,| \lesssim c_\epsilon \mathtt{d}(w,w')\,\mathtt{d}(w',z)^2 \le \mathtt{d}(w,w')\,\mathtt{d}(w',z) \lesssim \mathtt{d}(w,w')^2 + \mathtt{d}(w,w')\,\mathtt{d}(w,z),$$

which establishes (5.5).

The near symmetry of the Cauchy kernel (up to negligeable errors) was essential in the treatment of the Cauchy-Szegő projection in the case of smooth domains, as in [KS-2]. Only a vestige of this fact remains

in our case and it is given as follows. We recall that $|g_\epsilon(w,z)| \approx |g_\epsilon(z,w)| \approx \mathtt{d}(w,z)^2$.

Lemma 21. *For every $\epsilon > 0$ there is an $s(\epsilon) > 0$, so that if $s \leq s(\epsilon)$ then*

$$|g_\epsilon(w,z) - \overline{g_\epsilon}(z,w)| \lesssim \epsilon\, \mathtt{d}(w,z)^2 \tag{5.6}$$

whenever $\mathtt{d}(w,z) \lesssim s$.

The proof of (5.6) uses the modulus of continuity of the second derivatives of ρ and is essentially given in [LS-2, Section 2.3]. It is shown there that the left-hand side of (5.6) is majorized by $c_0\epsilon\,|w-z|^2$ if $|w-z| \leq \delta(\epsilon)$. Since $|w-z| \leq c_1\mathtt{d}(w,z)$, we obtain (5.6) when we take $s(\epsilon) = c'\delta(\epsilon)$ for an appropriate constant c'.

Proof of Proposition 18. We begin by describing the truncations of the operators that we will deal with. If $\mathcal{C}_\epsilon$ is our Cauchy transform, then for $s > 0$ its truncated version $\mathcal{C}_\epsilon^s$ is defined by

$$\mathcal{C}_\epsilon^s(f)(z) = \mathcal{C}_\epsilon\big(f(\cdot)\,\chi_s(\cdot,z)\big)(z), \quad z \in bD. \tag{5.7}$$

Here $\chi_s(w,z)$ is the symmetrized version of the cut-off function given in the proof of Lemma 12. It is defined by

$$\chi_s(w,z) = \widetilde{\chi}_{s,w}(z)\,\widetilde{\chi}_{s,z}(w),$$

where we recall that

$$\widetilde{\chi}_{s,w}(z) = \chi\left(\frac{\mathrm{Im}\langle\partial\rho(w), w-z\rangle}{c\,s^2} + i\,\frac{|w-z|^2}{c\,s^2}\right)$$

with $\chi(u+iv)$ as in the proof of Lemma 12.

Instead of dealing directly with $\mathcal{C}_\epsilon$ and its truncation $\mathcal{C}_\epsilon^s$, we work with $\mathcal{C}_\epsilon^\sharp$ and its corresponding truncation $\mathcal{C}_\epsilon^{\sharp,s}$. We do so because identities (4.13) and (4.14) make the choice of the formal adjoint of $\mathcal{C}_\epsilon^\sharp$ obvious, as opposed to the situation for $\mathcal{C}_\epsilon$. Now, by what has been said above, the truncation $\mathcal{C}_\epsilon^{\sharp,s}$ is defined by

$$\mathcal{C}_\epsilon^{\sharp,s}(f)(z) = \mathcal{C}_\epsilon^\sharp\big(f(\cdot)\,\chi_s(\cdot,z)\big)(z), \quad z \in bD\,.$$

We observe that the kernel of $\mathcal{C}_\epsilon^{\sharp,s}$ is $g_\epsilon(w,z)^{-n}\chi_s(w,z)$, in the sense that

$$\mathcal{C}_\epsilon^{\sharp,s}(f)(z) = \int\limits_{bD} g_\epsilon(w,z)^{-n}\chi_s(w,z)f(w)\,d\lambda(w)\,,$$

whenever f satisfies the Hölder regularity (3.5) and $z \in bD$ is such that $f(z) = 0$; this is because a corresponding formula holds for $\mathcal{C}_\epsilon^\sharp$, see (4.9). Using the reasoning of Proposition 10 we can see that there is an

adjoint $(\mathcal{C}_\epsilon^{\sharp,s})^*$, defined on functions that satisfy the Hölder regularity (3.5), such that

$$\big((\mathcal{C}_\epsilon^{\sharp,s})^*(f_1), f_2\big) = \big(f_1, (\mathcal{C}_\epsilon^{\sharp,s})^* f_2\big)$$

whenever f_1 and f_2 are a pair of such functions. Moreover

$$(\mathcal{C}_\epsilon^{\sharp,s})^*(f)(z) = (\mathcal{C}_\epsilon^{\sharp})^*\big(f(\cdot)\,\chi_s(\cdot, z)\big)(z)\,,$$

and

$$(\mathcal{C}_\epsilon^{\sharp,s})^*(f)(z) = \int\limits_{bD} \overline{g_\epsilon}(z,w)^{-n}\chi_s(w,z) f(w)\, d\lambda(w)$$

with the second identity holding whenever $f(z) = 0$.

With these in place we turn to the proof of the analogue of (5.1), where the operator $\mathcal{C}_\epsilon^s$ is replaced by $\mathcal{C}_\epsilon^{\sharp,s}$. To simplify the notation we set

$$\mathcal{A}_\epsilon^s = \mathcal{C}_\epsilon^{\sharp,s} - (\mathcal{C}_\epsilon^{\sharp,s})^*.$$

We assert that the operator $\mathcal{A}_\epsilon^s$, initially defined on Hölder-type functions, extends to a bounded operator on $L^p(bD)$, with

$$\|\mathcal{A}_\epsilon^s\|_{L^p\to L^p} \lesssim \epsilon^{1/2} M_p \tag{5.8}$$

for any $1 < p < \infty$, as long as $s \leq s(\epsilon)$. (The relevance of the exponent 1/2 is discussed in Section 5.4 below, see Remark D there.)

Before coming to the proof of (5.8) let us see how this implies Proposition 18. To this end, we begin by writing

$$\mathcal{C}_\epsilon^s = \mathcal{C}_\epsilon^{\sharp,s} + \mathcal{R}_\epsilon^{\sharp,s}\,, \quad \text{and} \quad (\mathcal{C}_\epsilon^s)^* = (\mathcal{C}_\epsilon^{\sharp,s})^* + (\mathcal{R}_\epsilon^{\sharp,s})^*\,.$$

From these it follows that

$$\mathcal{C}_\epsilon^s - (\mathcal{C}_\epsilon^s)^* = \mathcal{A}_\epsilon^s + \mathcal{B}_\epsilon^s$$

with

$$\mathcal{B}_\epsilon^s = \mathcal{R}_\epsilon^{\sharp,s} - (\mathcal{R}_\epsilon^{\sharp,s})^*.$$

Now each of the components of the kernel of $\mathcal{B}_\epsilon^s$ is bounded by

$$\widetilde{c}_\epsilon \mathtt{d}(w,z)^{-2n+1}\chi_s(w,z),$$

see (4.3). From this and (2.17) it follows that the operator $\mathcal{T} = s^{-1}\mathcal{B}_\epsilon^s$ satisfies the estimates (4.21) and these in turn imply that

$$\|\mathcal{B}_\epsilon^s\|_{L^p\to L^p} \leq \widetilde{c}_\epsilon O(s)\,,$$

see (4.22). While the constant $\widetilde{c}_\epsilon$ may be large as $\epsilon \to 0$, we have that and $\widetilde{c}_\epsilon O(s) = O(\epsilon^{1/2})$, if s is sufficiently small. This proves Proposition 18 and as the argument in Section 5.2 also shows, gives the proof of Theorem 15, assuming the validity of (5.8).

5.4. **Kernel estimates and cancellation conditions of $\mathcal{A}^s_\epsilon$.** We will apply the $T(1)$ theorem to prove (5.8), and so we need to come to grips with two issues. First, there are the difference inequalities for the kernel of $\mathcal{A}^s_\epsilon$. Here, as opposed to the situation for $\mathcal{C}^\sharp = \mathcal{C}^\sharp_\epsilon$ and $(\mathcal{C}^\sharp)^* = (\mathcal{C}^\sharp_\epsilon)^*$ given in (4.18) and (4.19), we need to see an appropriate gain in ϵ (of order $\epsilon^{1/2}$). Second, for the cancellation properties, we will see a gain, in fact of order ϵ, beyond what holds for $\mathcal{C}^\sharp$ as stated in Proposition 13.

Let us now deal with the kernel of $\mathcal{A}^s_\epsilon$, which we denote by $A^s_\epsilon(w,z)$. According to our previous discussion it is

$$A^s_\epsilon(w,z) = \left(g_\epsilon(w,z)^{-n} - \overline{g_\epsilon}(z,w)^{-n}\right)\chi_s(w,z)\,.$$

Our objective is to prove the inequality:

$$|A^s_\epsilon(w,z) - A^s_\epsilon(w',z)| \lesssim \epsilon^{1/2}\frac{\mathtt{d}(w,w')^{1/2}}{\mathtt{d}(w,z)^{2n+1/2}} \quad \text{for any } s \le s(\epsilon), \tag{5.9}$$

under the proviso that

$$\mathtt{d}(w,z) \ge c\,\mathtt{d}(w,w')$$

for a sufficiently large constant c. We claim first that

$$|A^s_\epsilon(w,z)| \lesssim \epsilon\,\mathtt{d}(w,z)^{-2n}, \quad \text{if } s \le s(\epsilon). \tag{5.10}$$

In fact since $|g_\epsilon(w,z)| \approx |g_\epsilon(z,w)| \approx \mathtt{d}(w,z)^2$, we have

$$|g_\epsilon(w,z)^{-n} - \overline{g_\epsilon}(z,w)^{-n}| \lesssim \frac{|g_\epsilon(w,z) - \overline{g_\epsilon}(z,w)|}{|g_\epsilon(z,w)|^{n+1}}\,.$$

However the support of $\chi_s(w,z)$ restricts matters to w in $\mathbf{B}_s(z)$, and thus inequality (5.6) ensures that

$$|A^s_\epsilon(w,z)| \lesssim \epsilon\,\frac{\mathtt{d}(w,z)^2}{\mathtt{d}(w,z)^{2n+2}} = \epsilon\,\mathtt{d}(w,z)^{-2n}, \quad \text{when } \mathtt{d}(w,z) \le s,$$

and (5.10) is thus established. Note that under our assumptions $\mathtt{d}(w,z) \approx \mathtt{d}(w',z)$ and hence we also have

$$|A^s_\epsilon(w',z)| \lesssim \epsilon\,\mathtt{d}(w,z)^{-2n}, \quad \text{if } \ s \le s(\epsilon). \tag{5.11}$$

In particular this and (5.10) give

$$|A^s_\epsilon(w,z) - A^s_\epsilon(w',z)| \lesssim \epsilon\,\mathtt{d}(w,z)^{-2n} \quad \text{if } \ s \le s(\epsilon). \tag{5.12}$$

We next claim that we also have

$$|A^s_\epsilon(w,z) - A^s_\epsilon(w',z)| \lesssim \frac{\mathtt{d}(w,w')}{\mathtt{d}(w,z)^{2n+1}} \quad \text{as long as } \ s \le s(\epsilon). \tag{5.13}$$

This again follows by treating the constituents of $A^s_\epsilon(w,z) - A^s_\epsilon(w',z)$ separately. Specifically, we first rearrange the terms of $A^s_\epsilon(w,z) -$

$A^s_\epsilon(w',z)$ into three separate groups, namely *1.*, a group of terms involving the difference $g_\epsilon(w,z)-g_\epsilon(w',z)$; *2.*, terms that have $\overline{g_\epsilon}(z,w)-\overline{g_\epsilon}(z,w')$; and finally *3.*, terms with $\chi_s(w,z)-\chi_s(w',z)$. We then provide estimates for each such group.

To see that inequality (5.13) holds for the first group, we argue as in the proof of (4.18) and (4.19), except that now we invoke Lemma 20 which avoids the undesirable factor c_ϵ. The argument for the terms involving $\overline{g_\epsilon}(z,w)-\overline{g_\epsilon}(z,w')$ is similar, but here we do not need Lemma 20. Next, to treat the terms involving $\chi_s(w,z)-\chi_s(w',z)$ we note that this difference vanishes, unless the equivalent quantities

$$|g_\epsilon(w,z)| \approx |g_\epsilon(z,w)| \approx \mathtt{d}(w,z)^2$$

are comparable to s^2, and in such case this difference is easily seen to be dominated by a multiple of of

$$\frac{1}{s^2}\big(\mathtt{d}(w,w')^2+\mathtt{d}(w,w')\,\mathtt{d}(w',z)\big);$$

since $s\approx \mathtt{d}(w,z)\approx \mathtt{d}(w',z)$, here we have that

$$|\chi_s(w,z)-\chi_s(w',z)| \lesssim \frac{\mathtt{d}(w,w')}{\mathtt{d}(w,z)}.$$

Combining the above we obtain (5.13). Finally we combine (5.12) with (5.13) to get the desired regularity (5.9) via the geometric mean of these two inequalities. Furthermore, since

$$A^s_\epsilon(w,z) = -\,\overline{A^s_\epsilon}(z,w)\,,$$

the result (5.13) and therefore (5.9) with w and z interchanged also follows. This concludes the proof of the difference inequalities for $\mathcal{A}^s_\epsilon = \mathcal{C}^{\sharp,s}_\epsilon - (\mathcal{C}^{\sharp,s}_\epsilon)^*$.

The proof of the cancellation properties of $\mathcal{A}^s_\epsilon$ is based on the following lemma.

Lemma 22. *Suppose $f\in C^1(bD)$, with $|f(w)|\le 1$ for all $w\in bD$. Then for every $\epsilon>0$ there is an $s(\epsilon)>0$, so that if $s\le s(\epsilon)$ and $z\in bD$ we have*

$$|\mathcal{A}^s_\epsilon(f)(z)| \;\lesssim\; \epsilon+\epsilon\int\limits_{bD}|\nabla f(w)|\,\mathtt{d}(w,z)^{-2n+2}d\lambda(w)\,.$$

The implied constant is independent of ϵ, s (and f and z).

Proof. We begin by considering the kernel

$$\widetilde{A^s_\epsilon}(w,z) = \frac{1}{1-n}\big(g_\epsilon(w,z)^{-n+1}-\overline{g_\epsilon}(z,w)^{-n+1}\big)\,\chi_s(w,z)$$

for $w \in bD$ and z fixed in D. Our goal is to apply Lemma 9 to

$$F(w) = f(w)\widetilde{A^s_\epsilon}(w,z)$$

(and ρ the defining function of our domain). We claim that computing $dF \wedge (\partial\overline{\partial}\rho)^{n-1}$ for such an F will give rise to several terms. One of these will be $A^s_\epsilon(w,z)f(w)d\lambda(w)$ (whose integral over bD is precisely $\mathcal{A}^s_\epsilon(f)(z)$), while the integrals of the other terms will provide the required bound for $|\mathcal{A}^s_\epsilon(f)(z)|$.

To see this, we recall (4.12), which gives us

$$d_w g_\epsilon(w,z) = \partial\rho(w) + O_\epsilon(|w-z|)$$

and similarly,

$$d_w \overline{g_\epsilon}(z,w) = -\overline{\partial}\rho(w) + O_\epsilon(|w-z|).$$

Here O_ϵ indicates that the bound in the term $O_\epsilon(|w-z|)$ depends on ϵ. Indeed such bound is $C\cdot(1+c_\epsilon)$ with c_ϵ as in (5.4). Recalling that $j^*\partial\rho = -j^*\overline{\partial}\rho$, we get

$$d_w\widetilde{A^s_\epsilon}(w,z) = A^s_\epsilon(w,z)\partial\rho(w) + O_\epsilon\big(\mathtt{d}(w,z)^{-2n+1}\big)\chi_s(w,z) + \frac{\epsilon}{s^2}O\big(\mathtt{d}(w,z)^{-2n+2}\big)$$

because $|w-z| \lesssim \mathtt{d}(w,z)$ and $|g_\epsilon(w,z)| \approx |g_\epsilon(z,w)| \approx \mathtt{d}(w,z)^2$, where we recall that

$$g_\epsilon(w,z) = \langle \partial\rho(w), w-z\rangle - \frac{1}{2}\sum_{j,k}\tau^\epsilon_{jk}(w)\,(w_j-z_j)(w_k-z_k)$$

for w near z, see (2.2). In applying Lemma 9 with F as above we keep in mind that when d_w acts on $\chi_s(w,z)$ we obtain a term which is

$$O\left(|\widetilde{A^s_\epsilon}(w,z)\,f(w)|\,\frac{1}{s^2}\,\mathbb{1}_s(w,z)\right),$$

where $\mathbb{1}_s(w,z)$ is the characteristic function of $\mathbf{B}_s(z)$. As a result, we get that

$$\mathcal{A}^s_\epsilon(f)(z) \;=\; I + II + III\,, \quad z \in bD\,,$$

where

$$I = -\frac{1}{(2\pi i)^n}\int\limits_{bD}\widetilde{A^s_\epsilon}(w,z)\,df(w)\wedge j^*(\overline{\partial}\partial\rho(w)^{n-1})\,,$$

$$II = \int\limits_{\mathbf{B}_s(z)} O_\epsilon\big(\mathtt{d}(w,z)^{-2n+1}|f(w)|\big)$$

and

$$III = \int\limits_{\mathbf{B}_s(z)} O\left(\widetilde{A^s_\epsilon}(w,z)\frac{|f(w)|}{s^2}\right).$$

Now the argument given in (5.10) shows that

$$|\widetilde{A^s_\epsilon}(w,z)| \lesssim \epsilon\, \mathtt{d}(w,z)^{-2n+2}$$

as long as $\mathtt{d}(w,z) \leq s(\epsilon)$. Hence we have that

$$|I| \lesssim \epsilon \int\limits_{bD} |\nabla f(w)|\, \mathtt{d}(w,z)^{-2n+2} d\lambda(w).$$

The term II is bounded by

$$\int\limits_{bD} O_\epsilon\big(\mathtt{d}(w,z)^{-2n+1}\big)\, d\lambda(w)$$

and the term III is bounded by

$$\frac{\epsilon}{s^2} \int\limits_{\mathbf{B}_s(z)} O_\epsilon\big(\mathtt{d}(w,z)^{-2n+2}\big)\, d\lambda(w)\,.$$

These two terms are bounded respectively by $s \cdot O_\epsilon$ and $\epsilon \cdot O(1)$ in view of (2.17). This gives a bound which is a multiple of $\epsilon + sO_\epsilon \lesssim \epsilon$, if we take $s \leq s(\epsilon)$ with $s(\epsilon) = 1/O_\epsilon$. The lemma is therefore proved. □

We now apply Lemma 22 to prove the cancellation properties of the operator

$$\mathcal{A}^s_\epsilon = \mathcal{C}^{\sharp,s}_\epsilon - (\mathcal{C}^{\sharp,s}_\epsilon)^*\,.$$

We first have that the function $\mathcal{A}^s_\epsilon(1)$ is in $L^\infty(bD)$ (in fact it can be seen that it is a continuous function) and moreover

$$|\mathcal{A}^s_\epsilon(1)| \lesssim \epsilon\,, \quad \text{whenever} \;\; s \leq s(\epsilon)\,, \tag{5.14}$$

with the implicit constant independent of ϵ, which follows immediately from Lemma 22.

Since by its definition $(\mathcal{A}^s_\epsilon)^* = -\mathcal{A}^s_\epsilon$, we also have

$$|(\mathcal{A}^s_\epsilon)^*(1)| \lesssim \epsilon\,, \quad \text{whenever} \;\; s \leq s(\epsilon)\,. \tag{5.15}$$

Next we argue as in Section 4.5. If f_0 is a C^1-smooth function supported in a ball $\mathbf{B}_r(\hat{w})$ that satisfies $|f_0(w)| \leq 1$ and $|\nabla f_0(w)| \leq 1/r^2$, then we can show that

$$|\mathcal{A}^s_\epsilon(f_0)(\hat{w})| \lesssim \epsilon \quad \hat{w} \in bD. \tag{5.16}$$

Indeed by Lemma 22, it suffices to see that

$$\int |\nabla f_0(w)|\, \mathtt{d}(w,\hat{w})^{-2n+2}\, d\lambda(w) \;\lesssim 1\,.$$

But in fact our hypotheses on f_0 and (2.17) grant

$$\int |\nabla f_0(w)|\, \mathtt{d}(w,\hat{w})^{-2n+2}\, d\lambda(w) \le \frac{1}{r^2} \int\limits_{\mathbf{B}_r(\hat{w})} \mathtt{d}(w,\hat{w})^{-2n+2}\, d\lambda(w) \lesssim 1\,,$$

proving the assertion (5.16).

Altogether, the above shows that $T = \epsilon^{-1}\mathcal{A}^s_\epsilon$ satisfies the hypotheses of Lemma 12, and so its conclusion grants that

$$|\mathcal{A}^s_\epsilon(f)(z)| \lesssim \epsilon \text{ for all } z \in bD, \text{ and } \quad \|\mathcal{A}^s_\epsilon(f)\|_{L^2(bD)} \lesssim \epsilon r^{2n} \tag{5.17}$$

for $s \le s(\epsilon)$ and for any normalized bump function f associated to a ball $\mathbf{B}_r(\hat{w})$, whenever $\hat{w} \in bD$.

We recall the $T(1)$-theorem as was used in Section 4.5, after (4.19). We now invoke the following a uniform version of this theorem. We consider the operators $\epsilon^{-1/2}\mathcal{A}^s_\epsilon$. They satisfy kernel estimates and cancellation conditions that are uniform in ϵ, as can be seen by (5.9), (5.14), (5.15) and (5.17).

Under these uniform conditions one has that the operators $\epsilon^{-1/2}\mathcal{A}^s_\epsilon$ satisfy the bounds

$$\|\epsilon^{-1/2}\mathcal{A}^s_\epsilon\|_{L^p\to L^p} \lesssim M_p$$

with M_p independent of ϵ. This proves (5.8), thus Proposition 18, and therefore concludes the proof of the main result, Theorem 15.

Remark D. The argument that proves (5.9) also gives the inequality with the right-hand side of (5.9) replaced by

$$\epsilon^\beta \frac{\mathtt{d}(w,w')^{1-\beta}}{\mathtt{d}(w,z)^{2n+1-\beta}} \quad \text{for } \ 0 \le \beta \le 1\,.$$

Thus the power $\beta = 1/2$ grants the conclusions (5.9) and (5.8), but corresponding conclusions would also hold with $\epsilon^{1/2}$ replaced by ϵ^β for any $0 < \beta < 1$; on the other hand the cases $\beta = 0$ and $\beta = 1$ cannot be used, the latter because of the requirement (4.20) that α (that is, $1-\beta$) be positive.

5.5. **Proof of Proposition 19.** The proof follows the same lines as the argument above, and so we shall be brief. The only substantial difference is that we cannot avail ourselves of the gain in ϵ given by (5.6).

It is enough to show that the analogue of (5.3) holds, but with $\mathcal{C}^s_\epsilon$ replaced by $\mathcal{C}^{\sharp,s}_\epsilon$, since the difference $\mathcal{C}^s_\epsilon - \mathcal{C}^{\sharp,s}_\epsilon$ can be handled as we did in verifying the hypotheses (4.21) and (4.22) for the error term $\mathcal{B}^s_\epsilon$ in the proof of Proposition 18. First we have the analogue of (5.13) with $\Delta^s_\epsilon(w,z)$ replaced by the single terms $g_\epsilon(w,z)\chi_s(w,z)$ and $\overline{g_\epsilon}(z,w)\chi_s(w,z)$.

Next, the argument of Lemma 22 shows that

$$|\mathcal{C}_\epsilon^{\sharp,s}(f)(z)| \ \lesssim \ 1 \ + \int\limits_{w\in bD} |\nabla f(w)|\, \mathsf{d}(w,z)^{-2n+2}\, d\lambda(w)$$

whenever f is in $C^1(bD)$ and $|f(w)| \leq 1$ for any $w \in bD$.

Thus, by Lemma 12,

$$|\mathcal{C}_\epsilon^{\sharp,s}(f)(z)| \ \lesssim \ 1 \quad \text{and} \quad \|\mathcal{C}_\epsilon^{\sharp,s} f\|_{L^2(bD,d\lambda)} \lesssim 1$$

for any normalized bump function f. The same conclusion holds for $\mathcal{C}_\epsilon^{\sharp,s}$ replaced by $(\mathcal{C}_\epsilon^{\sharp,s})^*$.

At this point, an application of the $T(1)$ theorem, as before, completes the proof of the proposition.

6. The Cauchy-Szegő projection with respect to a more general measure

We now pass to the Cauchy-Szegő projection $\mathcal{S}_\omega$ defined with respect to a measure of the form $\omega\, d\sigma$ described at the beginning of Part II. Throughout this section we will write $\omega\, d\sigma = \varphi\, d\lambda$, where φ (equivalently, $\omega = \varphi\Lambda$) is a continuous strictly positive density on bD, and we will use the upper script † to designate the adjoint with respect to the inner product $(\cdot,\cdot)_\omega$ of $L^2(bD, \omega\, d\sigma) = L^2(bD, \varphi\, d\lambda)$.

Thus $\mathcal{S}_\omega$ is the orthogonal projection of L^2 onto $\mathcal{H}^2$ in the sense that

$$\mathcal{S}_\omega^\dagger = \mathcal{S}_\omega\,.$$

Our goal is to prove Theorem 16, that is the analogue of Theorem 15 in which $\mathcal{S}$ is replaced by $\mathcal{S}_\omega$.

6.1. **Outline of the proof of Theorem 16.** We begin by making two simple observations. First,

$$\mathcal{S}_\omega(I + \mathcal{C}_\epsilon^\dagger - \mathcal{C}_\epsilon) = \mathcal{C}_\epsilon\,. \tag{6.1}$$

This is the analogue of the identity *(c)* that followed from Proposition 17, and is proved in the same way as that result, when we use a corresponding version of Proposition 17 (also proved in [LS-5]), in which the Leray-Levi measure $d\lambda$ is now replaced by its weighted version $\varphi\, d\lambda$, which lead to the identities

$$\mathcal{S}_\omega\mathcal{C}_\epsilon = \mathcal{C}_\epsilon \quad \text{and} \quad \mathcal{C}_\epsilon\mathcal{S}_\omega = \mathcal{S}_\omega\,.$$

The second observation concerns any bounded operator T on $L^2(bD, \varphi\, d\lambda) \approx L^2(bD, d\lambda)$ and will be applied to $T = \mathcal{C}_\epsilon$. Let T^* denote the adjoint of

T with respect to the inner product

$$(f,g)_\lambda = \int\limits_{bD} f\,\overline{g}\,d\lambda\,,$$

and $T^\dagger$ its adjoint with respect to the inner product

$$(f,g)_\omega = \int\limits_{bD} f\,\overline{g}\;\varphi\,d\lambda\,.$$

Then we have that

$$T^\dagger = \varphi^{-1}\,T^*\,\varphi\,. \tag{6.2}$$

In fact, by the definition of $T^\dagger$ and T^* we have

$$\int\limits_{bD} f\,\overline{(T^\dagger g)}\,\varphi d\lambda = \int\limits_{bD}(Tf)\,\overline{g}\,\varphi d\lambda = \int\limits_{bD} f\,\overline{T^*(g\,\varphi)}d\lambda.$$

Since this holds for all $f \in L^2(bD)$, we get $T^*(g\,\varphi) = T^\dagger(g)\varphi$, for all $g \in L^2(bD)$, which is merely a restatement of (6.2).

The main thrust of the proof of Theorem 16 is carried by the following proposition. For any (complex-valued) continuous function φ on bD, consider the commutator

$$[\mathcal{C}^s_\epsilon, \varphi] = \mathcal{C}^s_\epsilon\,\varphi - \varphi\,\mathcal{C}^s_\epsilon\,.$$

Proposition 23. *Suppose φ and ϵ are given. There is $\overline{s}(\epsilon) > 0$ so that if $s \leq \overline{s}(\epsilon)$, then*

$$\|\,[\mathcal{C}^s_\epsilon, \varphi]\,\|_{L^p\to L^p} \lesssim \epsilon M_p\,, \quad \textit{for}\ \ 1 < p < \infty\,. \tag{6.3}$$

Here $M_p = p + p/(p-1)$ is as in Proposition 18. We prove Proposition 23 in Section 6.3 below, but now note that this assertion immediately implies the analogous statement with $\mathcal{C}^s_\epsilon$ replaced $(\mathcal{C}^s_\epsilon)^*$. Indeed,

$$[(\mathcal{C}^s_\epsilon)^*, \varphi] = (\mathcal{C}^s_\epsilon)^*\,\varphi - \varphi\,(\mathcal{C}^s_\epsilon)^* = \big(\overline{\varphi}\,\mathcal{C}^s_\epsilon - \mathcal{C}^s_\epsilon\,\overline{\varphi}\big)^* = -\big([\mathcal{C}^s_\epsilon, \overline{\varphi}]\big)^*.$$

Thus (6.3) (with φ replaced by $\overline{\varphi}$, and with p and M_p replaced by p' and $M_{p'} = M_p$ respectively, where $1/p + 1/p' = 1$) implies by duality that

$$\|\,[(\mathcal{C}^s_\epsilon)^*, \varphi]\,\|_{L^p\to L^p} \lesssim \epsilon\,M_p, \quad \text{for}\ \ 1 < p < \infty\,. \tag{6.4}$$

The significant consequence of (6.4) is the analogue of Proposition 18, in which $(\mathcal{C}^s_\epsilon)^*$ is replaced by $(\mathcal{C}^s_\epsilon)^\dagger$. Namely, for any $\epsilon > 0$, there is an $s(\epsilon) > 0$ so that $s \leq s(\epsilon)$ implies

$$\|\,\mathcal{C}^s_\epsilon - (\mathcal{C}^s_\epsilon)^\dagger\,\|_{L^p\to L^p} \lesssim \epsilon^{1/2}\,M_p, \quad \text{for}\ \ 1 < p < \infty\,. \tag{6.5}$$

To see why this is the case, recall by (6.2) that $(\mathcal{C}^s_\epsilon)^\dagger = \varphi^{-1}(\mathcal{C}^s_\epsilon)^*\varphi$. Thus, $\mathcal{C}^s_\epsilon - (\mathcal{C}^s_\epsilon)^\dagger = \mathcal{C}^s_\epsilon - \varphi^{-1}(\mathcal{C}^s_\epsilon)^*\varphi$. But

$$\varphi^{-1}(\mathcal{C}^s_\epsilon)^*\varphi - (\mathcal{C}^s_\epsilon)^* = \varphi^{-1}\big((\mathcal{C}^s_\epsilon)^*\varphi - \varphi(\mathcal{C}^s_\epsilon)^*\big)\,.$$

So

$$\mathcal{C}^s_\epsilon - (\mathcal{C}^s_\epsilon)^\dagger = \mathcal{C}^s_\epsilon - (\mathcal{C}^s_\epsilon)^* + \varphi^{-1}[\varphi, (\mathcal{C}^s_\epsilon)^*]\,.$$

Hence

$$\|\,\mathcal{C}^s_\epsilon - (\mathcal{C}^s_\epsilon)^\dagger\,\|_{L^p\to L^p} \le \|\,\mathcal{C}^s_\epsilon - (\mathcal{C}^s_\epsilon)^*\,\|_{L^p\to L^p} + \sup|\varphi^{-1}|\,\|\,[\mathcal{C}^s_\epsilon,\,\varphi]\,\|_{L^p\to L^p}\,.$$

By Proposition 18 and (6.4) the righthand side of the above is majorized by a multiple of $\epsilon^{1/2}M_p + \epsilon M_p$, which gives us (6.5).

With (6.5) in hand, the proof of Theorem 16 follows the reasoning given in Section 5.2 after Proposition 18. Indeed, by (6.1) we get the analogue of (5.2), namely

$$\mathcal{C}_\epsilon + \mathcal{S}_\omega(\mathcal{R}^s_\epsilon)^\dagger - \mathcal{S}_\omega\mathcal{R}^s_\epsilon = \mathcal{S}_\omega\big(I + \mathcal{C}^s_\epsilon - (\mathcal{C}^s_\epsilon)^\dagger\big)\,,$$

since $\mathcal{C}_\epsilon = \mathcal{C}^s_\epsilon + \mathcal{R}^s_\epsilon$. The rest of the proof of of Theorem 16 is then an almost word-for-word repetition of the argument after (5.2). Here we invoke (6.5), which guarantees that $I + \mathcal{C}^s_\epsilon - (\mathcal{C}^s_\epsilon)^\dagger$ is invertible in L^p when ϵ and s are taken to be sufficiently small. With this, Theorem 16 is proved (assuming the validity of Proposition 23).

6.2. **A decomposition lemma.** The proof of Proposition 23 requires the following decomposition lemma, valid for a general class of operators. We consider a partition of $\mathbb{C}^n = \mathbb{R}^{2n}$ into disjoint cubes of side-length γ, given by

$$\mathbb{C}^n = \cup_{k\in\mathbb{Z}^{2n}} Q^\gamma_k\,.$$

When $\gamma = 1$, we take $Q^1_k = Q^1_0 + k$, where k ranges over the lattice points $\mathbb{Z}^{2n}$ of $\mathbb{R}^{2n}$, and Q^1_0 is the unit cube in $\mathbb{R}^{2n}$ with center at the origin. For general $\gamma > 0$, we set $Q^\gamma_k = \gamma\, Q^1_k$. We say that the cube Q^γ_k *touches the cube* Q^γ_j if $\overline{Q^\gamma_k}$ and $\overline{Q^\gamma_j}$ have non-empty intersection; equivalently, if $dist(Q^\gamma_j, Q^\gamma_k) \le \gamma$ (with *dist* the Euclidean distance). One notes that for each k there are exactly $N = 3^{2n}$ cubes Q^γ_j that touch Q^γ_k. These facts are easily verified by inspection when one scales back to the case $\gamma = 1$.

We now revert to our domain $D \subset \mathbb{C}^n$. For a fixed $\gamma > 0$ we write $\mathbb{1}_k$ for the characteristic function of $Q^\gamma_k \cap bD$. Thus what has been said above implies that

$$\sum_{j\in\mathbb{Z}^{2n}} \mathbb{1}_j = 1 \quad \text{on} \quad bD\,, \quad \text{and}$$

$$\sum_k \Big(\sum_{Q_j^\gamma \text{ touches } Q_k^\gamma} \mathbb{1}_j \Big) = N \quad \text{on} \quad bD\,. \tag{6.6}$$

Lemma 24. *Suppose T is a bounded operator on $L^p(bD)$. Fix $\gamma > 0$ and assume*

(i) $\mathbb{1}_k T \mathbb{1}_j = 0,$ *if the the cubes Q_j^γ and Q_k^γ do not touch.*

(ii) $\|\mathbb{1}_k T \mathbb{1}_j\|_{L^p\to L^p} \leq A,$ *otherwise.*

Then

$$\|T\|_{L^p\to L^p} \leq A\,N\,.$$

Here $\mathbb{1}_k T \mathbb{1}_j(f) = \mathbb{1}_k T(f\mathbb{1}_j)$.

Proof. We begin by observing that

$$\int_{bD} |Tf|^p = \sum_k \int_{Q_k^\gamma\cap bD} |Tf|^p\,.$$

However,

$$\int_{Q_k^\gamma\cap bD} |Tf|^p = \int_{bD} \mathbb{1}_k\,|Tf|^p\,,$$

while

$$\mathbb{1}_k Tf = \mathbb{1}_k \sum_j T(f\mathbb{1}_j)\,,$$

since $\sum \mathbb{1}_j = 1$ on bD. Now by (i), $\mathbb{1}_k T(f\mathbb{1}_j) = 0$ unless Q_k^γ touches Q_j^γ, and so at most N terms in the above sum are non-zero. Moreover we always have $\big|\sum a_j\big|^p \leq N^{p-1}|\sum |a_j|^p$, if at most N of the a_j are non-zero, as Hölder inequality shows. Thus

$$\mathbb{1}_k|Tf|^p = \mathbb{1}_k\Big|\sum_j T(f\mathbb{1}_j)\Big|^p \leq N^{p-1}\mathbb{1}_k^p \sum_{Q_j^\gamma \text{ touches } Q_k^\gamma} |T(f\mathbb{1}_j)|^p\,.$$

Hence

$$\int_{bD} |Tf|^p \;\leq\; N^{p-1} \sum_{Q_j^\gamma \text{ touches } Q_k^\gamma} \int_{bD} |\mathbb{1}_k T(f\mathbb{1}_j)|^p \;\leq$$

$$\leq N^{p-1}A^p \sum_{Q_j^\gamma \text{ touches } Q_k^\gamma} \int_{bD} |f|^p|\mathbb{1}_j|^p\,,$$

and by (6.6) and hypothesis *(ii)* the latter equals

$$N^{p-1}NA^p\sum_j\int_{bD}|f|^p\mathbb{1}_j \;=\; N^pA^p\int_{bD}|f|^p.$$

The lemma is therefore proved. □

6.3. **Completion of the proof of Theorem 16.** We will apply the lemma above to the operator $T=[\mathcal{C}^s_\epsilon,\varphi]$. So given a continuous function φ and an $\epsilon>0$, we will show that we can pick a $\gamma>0$ and an $s(\epsilon)>0$, so that if $s\le s(\epsilon)$, then T satisfies the hypotheses *(i)* and *(ii)* of Lemma 24, with $A\approx\epsilon$.

To begin with, we note that if z is not in the support of f, then

$$\mathcal{C}^s_\epsilon(f)(z)=\int_{w\in bD}C_\epsilon(w,z)\chi_s(w,z)f(w)\,d\lambda(w), \tag{6.7}$$

where $C_\epsilon(w,z)$ is the kernel of $\mathcal{C}_\epsilon$ (see (4.2) and recall that $\mathcal{C}_\epsilon$ is denoted $\mathcal{C}$ in Section 4), and $\chi_s(w,z)$ is the symmetrized cut-off function that occurred in (5.7).

Next we consider $\mathbb{1}_kT\mathbb{1}_j$ and observe that with the right choice of γ, the truncated operator $\mathbb{1}_kT\mathbb{1}_j$ vanishes whenever Q^γ_j and Q^γ_k do not touch. Indeed when Q^γ_j and Q^γ_k do not touch, if $z\in Q^\gamma_k$ then z does not lie in the support of $f\mathbb{1}_j$. Hence by (6.7) we have

$$\mathbb{1}_k(z)\,\mathcal{C}^s_\epsilon(f\mathbb{1}_j)(z)=\mathbb{1}_k(z)\int_{bD}C_\epsilon(w,z)\chi_s(w,z)f(w)\mathbb{1}_j(w)\,d\lambda(w)\,.$$

There is a similar formula with f replaced by $f\varphi$. Thus for either $\mathbb{1}_k(z)\varphi(z)\mathcal{C}^s_\epsilon(f\mathbb{1}_j)(z)$ or $\mathbb{1}_k(z)\mathcal{C}^s_\epsilon(f\mathbb{1}_j\varphi)(z)$ not to vanish, there must be a $z\in Q^\gamma_k$ and a $w\in Q^\gamma_j$ with $dist(w,z)\le\gamma$, (given the support condition of $\chi_s(w,z)$).

But since Q^γ_j and Q^γ_k d not touch, necessarily $|z-w|>\gamma$, and this combined with the inequality $dist(w,z)=|w-z|\le c\mathtt{d}(w,z)$, (see (2.13)) give a contradiction to $\gamma\ge cs$. So we fix γ with $\gamma=cs$, then this guarantees hypothesis *(i)* of the lemma, for our choice of T. Next we prove that if γ and s are chosen sufficiently small, then hypothesis *(ii)* is satisfied with $A\approx\epsilon$. To see this we let z_k denote the center of the cube Q^γ_k. Then $z\in Q^\gamma_k$ implies that $|z-z_k|\le\sqrt{k}\gamma$. Also if Q^γ_k touches Q^γ_j, then $|w-z_k|\le(\sqrt{k}+1)\gamma$ whenever $w\in Q^\gamma_j$. As a result, the uniform continuity of φ grants that

$$\sup_{z\in Q^\gamma_k}|\varphi(z)-\varphi(z_k)|\le\epsilon\qquad\text{and}\qquad\sup_{w\in Q^\gamma_j}|\varphi(w)-\varphi(z_k)|\le\epsilon\,, \tag{6.8}$$

as long as γ is made sufficiently small (in terms of ϵ).

Now write $\varphi = \varphi_k + \psi_k$, where $\varphi_k(z) = \varphi(z) - \varphi(z_k)$, and $\psi_k(z) = \varphi(z_k)$, for all z. Then $[\mathcal{C}^s_\epsilon, \varphi] = [\mathcal{C}^s_\epsilon, \varphi_k] + [\mathcal{C}^s_\epsilon, \psi_k]$. But ψ_k is a constant, so $[\mathcal{C}^s_\epsilon, \psi_k] = 0$, and hence

$$\mathbb{1}_k T \mathbb{1}_j = \mathbb{1}_k [\mathcal{C}^s_\epsilon, \varphi]\mathbb{1}_j = \mathbb{1}_k [\mathcal{C}^s_\epsilon, \varphi_k]\mathbb{1}_j = \mathbb{1}_k \mathcal{C}^s_\epsilon(\varphi_k \mathbb{1}_j) - \mathbb{1}_k \varphi_k \, \mathcal{C}^s_\epsilon \mathbb{1}_j \, .$$

However $\sup\limits_{z\in bD} |(\varphi_k \mathbb{1}_j)(w)| \le \sup\limits_{z\in Q^\gamma_k} |\varphi(z) - \varphi(z_k)|$. Therefore (6.8) gives

$$\|\mathbb{1}_k T \mathbb{1}_j f\|_{L^p} \;\le\; 2\epsilon \|\, \mathcal{C}^s_\epsilon \,\|_{L^p\to L^p} \|f\|_{L^p} .$$

Thus hypothesis *(ii)* holds with $A = 2\epsilon \|\, \mathcal{C}^s_\epsilon \,\|_{L^p\to L^p}$. Finally, Proposition 19 ensures that $\|\mathcal{C}^s_\epsilon\|_{L^p\to L^p} \lesssim M_p$, if s is sufficiently small. Thus Lemma 24 implies $\|[\mathcal{C}^s_\epsilon, \varphi]\|_{L^p\to L^p} \lesssim \epsilon M_p$, thus proving Proposition 23. The proof of Theorem 16 is thus now concluded.

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